# An Archetypal Graphic Model in the *Conics* of Apollonius

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## Introduction

In the ancient world, and therefore well before the codification of the double orthogonal projection method developed by Gaspard Monge [Monge 1799], it was possible to represent three-dimensional space as faithfully as methods of a projective nature allow today. These representations, while not being documented by means of graphics, are in fact documented thanks to the descriptions obtained from texts, such as that of the *Conics* of Apollonius of Perga.
The purpose of this short essay is to demonstrate, with an example, how these accurate descriptions of plane sections of solid figures, executed in such a way as to preserve the true shape of the elements, are able to allow the reconstruction in space of the architecture of geometric shapes and relationships and to perform verifications by means of graphic calculation.
*Proposition 13* in *Book I* of the *Conics* provides a good example of this ancient method, which has been passed on through the centuries leaving deep traces even today in the pages of modern stereotomy. The two-dimensional figures, linked together by an artifice different, but no less effective, from the use of lines connecting projections of the same point, were intended as a three-dimensional model and ideally reassembled in space, and we can imagine them as pages of a pop-up book. This usage is suggested, for example, by the term "*subiacens*" used in the last propositions of *Book I* to denote the plane section on which the complex edifice of the geometry of the cone and its curves is constructed.

*This article was written upon invitation to frame the topic, not submitted to anonymous review, published under the editorial director's responsibility.*

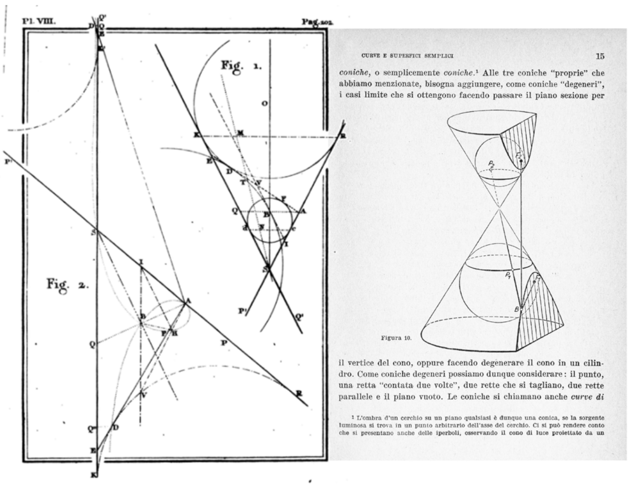



*Fig. 1. Left: the Dandelin-Quetelet construction, for the cases of the parabola and the ellipse, as it appears for the first time in the memoir presented by Germinal Pierre Dandelin to the Belgian Academy in 1822.*
*Right: the same construction for the case of the hyperbola, illustrated by David Hilbert about 100 years later.*

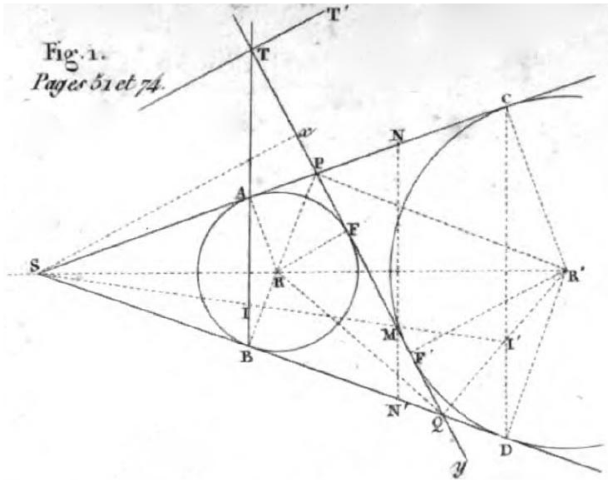



*Fig. 2. Detail of the supplementary table (A) of the second edition of Jean-Nicholas Pierre Hachette's Traité de Géométrie descriptive [1828], where Dandelin's theorem was first proposed in a course on Descriptive Geometry. The representation of the sectioned cone and of the two spheres inscribed within it, to study the case of the ellipse, is very similar to that of Dandelin. However, Hachette also denotes, by the symbol T, the projection of the line of intersection of the plane of section with the plane that contains the circle of contact of one of the spheres with the cone and identifies, in this line, one of the two directrices of the curve. Hachette recalls that this additional property of Dandelin's "architecture" was pointed out to him by Mr. Blanchet, adjunct professor at the Royal Colleges of Paris, in 1826 [Hachette 1828, pp. 51, 52, 74].*

Finally, it must be noted that if the editions of Apollonius's treatise, like those of other works dealing with forms generated in three-dimensional space, had contained more faithful illustrations, these would benefit, and greatly, the reading of the text as well as its comprehension. These illustrations should not merely give a vague idea of the configurations imagined by the author, but could be used as testimony to a way of thinking and doing geometry, and also as tools for verifying relationships and operational practices.
Those who deal with Descriptive Geometry are inevitably led to study conic sections, not only because these curves arise whenever a plane "meets" [1] a cone or a cylinder, but also because they generate and are generated by surfaces such as the sphere, the ellipsoid, the hyperboloid, the paraboloid, and the hyperbolic paraboloid, all of which are of extreme interest to the theory as well as to the applications of this science.
Germinal Pierre Dandelin and his friend Adolphe Quetelet devised a simple construction that makes it possible to demonstrate how the section of a right circular cone is a parabola, a hyperbola, or an ellipse, when certain properties of these curves are given as known, or to prove the properties themselves by recognizing, in the plane section of a cone, a parabola, a hyperbola or an ellipse, as the case may be [2]. And this famous theorem, at least since David Hilbert used it in his Intuitive Geometry [Hilbert, Cohn-Vossen 1972, pp. 12-19], is an essential teaching tool (figs. 1, 2) [3].
But what was the structure that allowed Apollonius of Perga [4] to define conics as sections of an oblique or, as we say today, quadric cone, and not a right circular cone as in the abovementioned theorem?
In the Dandelin-Quetelet theorem [Dandelin 1822], the image, whether drawn or mental, plays an essential role: it makes manifest the abstract reasoning that proves the theorem. And if the image can do so much, in that theorem, it is because, thanks to Descriptive Geometry, that solid construction can be projected in the pages of a book and can provide, to those who wish to reproduce it, the tool for a quasi-experimental verification.

In other words, the image is, in itself, an existential proof [5], even if it does not have the force of a logical demonstration. The image, moreover, has a heuristic value: it helps to 'find' the truth, it suggests the truth, and therefore hints at what the thinking, between intuitions and deductions, of its inventor might have been.
So, if what I have said can hold for the nineteenth-century Dandelin-Quetelet theorem, why should it not hold for Apollonius's theorems? There is only one not small difference between the two ways of thinking in geometry that lies, precisely, in the capacity to represent it.
This is because, in the early nineteenth century, there was total control of three-dimensional forms, thanks to the contribution of Gaspard Monge and of those who preceded him in the modern age, while little or nothing is known of scientific representation in the time of Apollonius. In fact, all that remains of the graphic models from that remote time are the engravings on the stone of ancient building sites, which are fragments of orthographic projections, as in the case of the tympanum of the Pantheon that is seen in Rome on the pavement in front of the Mausoleum of Augustus, or nomograms, such as those on the Temple of Apollo at Didyma [6].

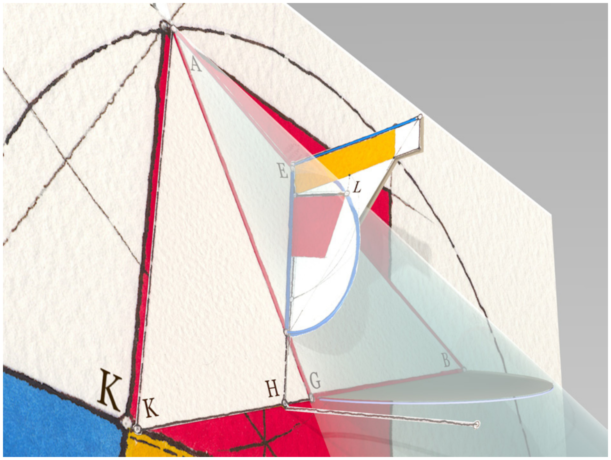


*Fig. 3. Synthesis of the archetypal model described by Apollonius: against the background of the nomogram that allows the latus rectum to be calculated graphically, the planes of the axial triangle and of the section ellipse open up, as in the pages of a pop-up book, capable of evoking the mental image of the cone represented here in transparency.*

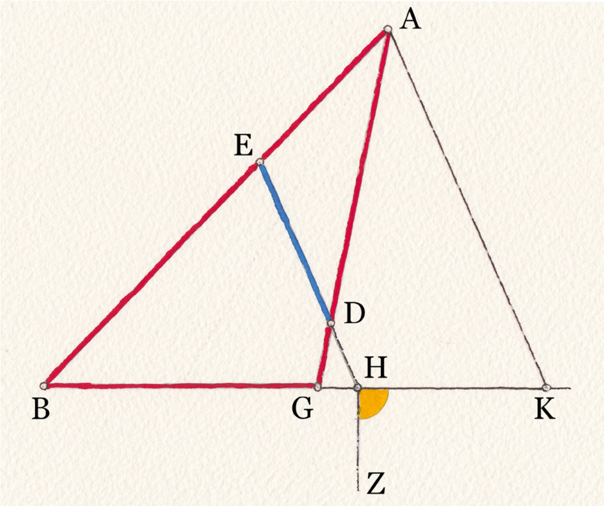
## The drawing of space in *Book I* of the *Conics*

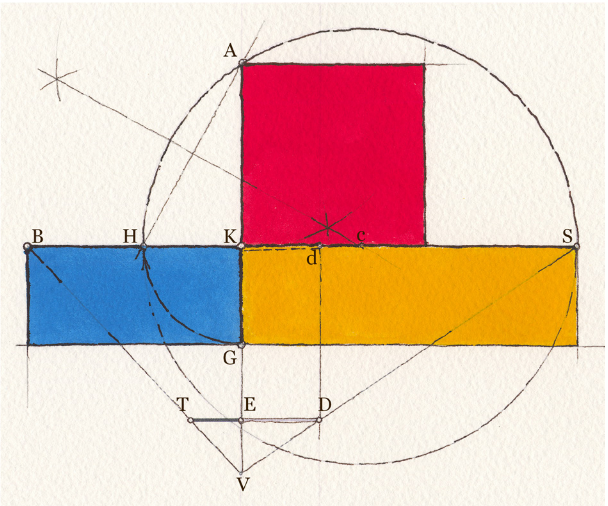
The treatise on conics not only consists of a set of logical deductions capable of textually describing the properties of the plane sections of the cone, but also describes, just as accurately, very elaborate geometrical structures that are functional to these deductions [7]. To define the ellipse, the parabola, and the hyperbola in *Propositions 11*, *12*, and *13* of *Book I*, Apollonius associates the double-napped oblique cone, which he considers for this purpose, with four planes: one as the base of the cone, a second that intersects it passing through the vertex, a third that cuts the cone and supports the conic section considered and a fourth, parallel to the base. Within these planes, there are circumferences and line segments all related to each other by relationships deduced from each other through logical steps that recall various theorems of Euclid. As a whole, these figures and their geometric relationships constitute a model of the cone and its section (fig. 3) [8].
Paul Ver Eecke observes that reading these propositions is "quite arduous" [Apollonius Pergæi 1923, p. XIII] [9] and many attempts have been made to transpose the reasoning into terms more accessible to us, either by using modern symbolic notation or through metaphors [Flaumenhaft 2013, pp. XIII-XXX]. But a faithful graphic representation is the most direct way to enter into the logic and thought of Apollonius.
Following the original text to the letter, this model consists of three drawings:

- the first is the "axial triangle" [10], described in points 3 and 5 of the *Definitions* as a section of the cone with a plane passing through the vertex and through the center of the circular section that forms its base. This triangle is represented in true shape and can evoke the idea of a solid finite cone, as in *Definition 2* [Apollonius Pergæi 1891, p. 7] [11]. The drawing is completed by the intersection of the axial triangle with the plane of the section that generates the conic (fig. 4);
- the second drawing is a nomogram [Cassinis 1928], which Apollonius defines with a relation between elements belonging to the axial triangle and two line segments belonging to the plane of the conic section: the *latus transversum* [12], that

*Fig. 4. The first of the drawings described in Proposition 13: the orthographic projection of the cone. Note that the plane that the axial triangle belongs to does not coincide with the apparent contour of the cone with respect to the direction perpendicular to that plane. Which confirms the non-projective character of this image.*

*Fig. 5. The second drawing: a nomogram which allows us to calculate the length of the latus rectum from the segments that are measured on the axial triangle: AK, KB and KG.*




is, the diameter (*Definition 4*), and the *latus rectum* [Apollonius Pergæi 1891, p. 43], whose length can be calculated graphically, thanks precisely to the nomogram (fig. 5);
- finally, the third drawing is the true shape of the conic, which can be drawn thanks to the diameter and the *latus rectum* after having measured its length (fig. 6).

In this short essay, it is not possible to examine all the three-dimensional graphical constructions described by Apollonius in *Book I* of the *Conics*; we will, therefore, limit ourselves to the construction of the conic in the case in which the cutting plane meets two sides of the axial triangle and, therefore, all the generatrices of the cone, which is the case of the ellipse, and which, in my opinion, is also the simplest. This examination will be conducted from a particular point of view, that of the architect who admires a building through its representation, as though it were a project.

## The drawing of the cone and the ellipse as geometric locus

*Proposition 13* begins with a discursive description of the cone and of the relationships that relate it to its section. The text between quotation marks ("…") is the original text; I have added, between square brackets ([…]), the letters that allow us to reconnect the text with the drawing. Apollonius does not do so, which demonstrates the literary nature of this description. We are in the context of an *ekphrasis* (figs. 4, 5) [13].
"If a cone is cut by a plane [*ABG*] through its axis, and also cut by another plane [*EZH*] meeting both sides [*AB* and *AG*] of this axial triangle (fig. 4), but is neither parallel to the base of the cone nor parallel to the opposite [section] [14]; and if the plane the base of the cone belongs to and the cutting plane meet in a straight line [*ZH*] perpendicular to the base [*BG*] of the axial triangle or to it continued [15], then any straight line [*LM*] drawn from the conic section to the diameter of the same [*ED*], so that it is parallel to the intersection [*ZH*] of the two planes [16], squared, will be equivalent to some rectangle [*EMXO*], applied to a straight line [*ET*] [17] to which the diameter [*ED*] of the section has the same ratio as the square of the segment [*AK*] drawn, parallel to the diameter, from the cone's vertex to the triangle's base, with respect to the rectangle contained between the segments [*KB*

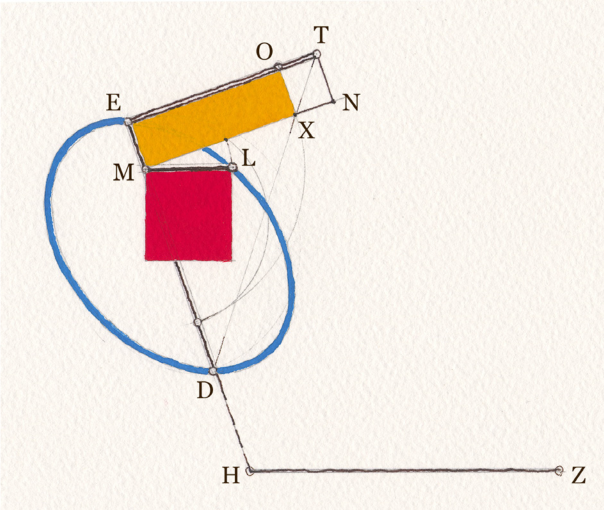



*Fig. 6. The third drawing described in the enunciation of Proposition 13. The length of the diameter ED is obtained from the first drawing. The length of the latus rectum ET is calculated graphically by means of the nomogram (fig. 5). By varying the position of the point M on the diameter ED so that the square of LM and the rectangle EX are equivalent, the point L describes the ellipse. The segment EO will always be smaller than the latus rectum ET, which therefore assumes the role of a parameter. And the segment OT is what the parameter lacks to reach the length of ET. 'Lack' is indeed, in Greek, the name of the ellipse.*

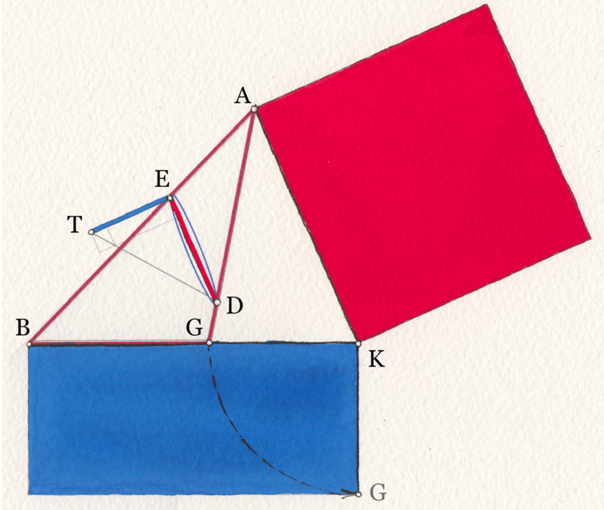



*Fig. 7. In this drawing, the cutting plane, which contains the latus rectum ET, has been superimposed on the axial triangle, leaving the position of the latus transversum ED unchanged. Here the colors highlight the corresponding elements in the relation (2).*

and *KG*] cut off on the base of the triangle by the aforementioned segment" [Apollonius Pergaei 1891, pp. 48, 49, translation by the author] (fig. 6).
Here the reading is particularly difficult precisely because ekphrasis lends itself well to describing shapes, but is not suitable for talking about relationships between geometric figures. Summarily, Apollonius, referring to the two drawings above, states that (fig. 6):

$$LM^2 = (EM \times MX) \qquad (1)$$

and that (fig. 5):

$$ED : ET = AK^2 : (KB \times KG) \qquad (2)$$

an expression that establishes the length of the *latus rectum ET* in relation to the axial triangle *ABG* (fig. 7).
It is now a matter of tying together the two previous relations, thus the enunciation goes on to define the rectangle (*EMXO*) in relation to the *latus rectum ET* (fig. 6): "And this rectangle applied to the *latus rectum* [*ET*] will have, as its breadth, the segment [*EM*] cut off on the diameter beginning from the section's vertex [*E*] to the point [*M*], in which the diameter is cut by the straight line [*LM*] drawn from the section to the diameter, while its area [*EMNT*] will be diminished by a figure [*OXNT*] similar and similarly situated to the rectangle contained by the diameter and the *latus rectum*. [...] Let such a section be called an *ellipse*" (fig. 6).
Therefore, as we shall see (fig. 8), the ellipse generated by that cutting plane can be drawn by arbitrarily choosing any point *M* of the diameter to then construct the geometric locus described by the relation (1).
At this point, all that remains is to graphically calculate the length of the *latus rectum* by means of the relation (2), in which all quantities, except for the unknown *ET*, can be measured on the drawing of the axial triangle (fig. 4) [18].
The relation (2) is given as true, *a priori*. Its validity is proven in the subsequent demonstration, which also justifies the relation (1).

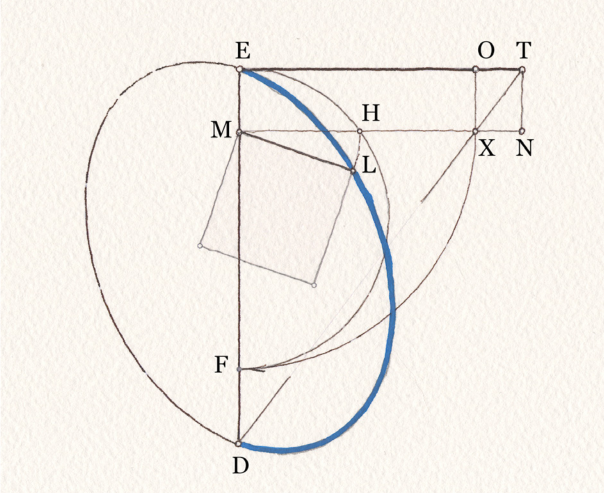



*Fig. 8. The construction of the ellipse given the diameter (latus transversum) ED and the latus rectum ET.*

## The nomogram that measures the *latus rectum*

The demonstration is developed in a dozen steps, for which I refer to the original text [Apollonius Pergaei 1891, pp. 48-53] [19], since what we are interested in here is the three-dimensional model as a whole, evoked by the drawings present in figures 4 and 8 connected by the nomogram in figure 5, which links the orthographic projection of the cone to the true shape of the section by means of the *latus rectum ET*.
The correspondence between the equation (2) and the drawing in figure 5 is direct: assuming that the yellow rectangle *GS* [20] and the red square $AK^2$ are equivalent by construction (see Note 18), the yellow rectangle is to the blue rectangle as the diameter *ED* is to the *latus rectum ET*. Therefore:

1. on the base of the axial triangle (fig. 4), the line segments *KB* and *KG* are measured, and a rectangle (in blue in fig. 5) is constructed, in which the two sides are equal in length to the abovementioned segments;
2. a segment *KH* equal in length to *KG* is marked on *KB*;
3. a segment *KA*, perpendicular to *KB* and equal in length to *AK*, is raised on *K*;
4. the circumference that has its center on the prolongation of *KB* and passes through *H* and *A* is constructed. This circumference cuts the prolongation of *KB* in *S*, and thus the rectangle having *KG* and *KS* as its sides has an area equal to that of the square of *AK*. In fact [Euclid 1970, VI, 13, p. 379], *AK* is the mean proportional between *HK*, which is equal to *KG* by construction, and *KS*, so that:

$$KG : AK = AK : KS$$

And that is:

$$AK^2 = KG \times KS$$

Therefore, to construct the *latus rectum ET* it is only necessary to find a line segment that satisfies the proportion:

$$ED : ET = KS : KB$$

In fact, rectangles of the same height are to one another as their bases [Euclid 1970, VI, 1, pp. 361-363], and thus the base *KS* of the yellow rectangle, which is equivalent to the red square ($AK^2$), is to the base of the blue rectangle *KB*, as the diameter *ED* is to the *latus rectum ET*.
It will suffice to put *ED* in relation with *KS*, for example, by constructing any triangle (such as *BSV* in fig. 5) and then cutting it with a straight line parallel to *BS*, so that the resulting segments *ET*, *ED*, have the desired relationships.
Having obtained the *latus rectum ET* as above, one can generate the curve on the cutting plane in true shape (fig. 8).

## The drawing of the ellipse

1. The right triangle *EDT*, which has the diameter *ED* and the *latus rectum ET* as its catheti, is constructed;
2. any point *M* is chosen on the diameter *ED* and *MN* is drawn through *M*, parallel to *ET*;
3. this straight line meets the hypotenuse of *EDT* at point *X*;
4. with a compass, *MX* is marked on the diameter as *MF*;
5. the semicircle whose diameter is *EF* is constructed and cuts the straight line *MN* at point *H*;
6. the chord conjugate to the diameter *ED* which is parallel to the intersection *ZH* of the cutting plane with the base of the cone, is drawn through point *M*;
7. with a compass, *MH* is marked on the abovementioned chord as *ML*.
8. *L* is a point of the ellipse, in fact:

$$MH^2 = EM \times MF$$

But given that $MH = ML$ and $MF = MX$

$$LM^2 = EM \times MX$$

Obviously, varying the choice of *M*, all the points on the curve are obtained (fig. 8).

## Conclusions on the purpose of this study

I realize that the fragmentary reading of the text by Apollonius I have referred to, which has, furthermore, been stripped of all or most of the logical steps of the demonstrations, may appear detrimental to the author's greatness and therefore unacceptable. But it must be remembered that this study does not look at mathematics, but at architecture and drawing.
How did the ancients realize the grandiose buildings they have left us, intended as constructions of stone, but also understood as constructions of thought, without the support of scientific representation as we know it today? Hinting at a possible answer is the purpose of this study to show that:

- when Apollonius of Perga deals with solid figures the text can be read as the *ekphrasis* of a three-dimensional model capable of representing space not only allusively but with operational capabilities; this analysis can be extended to many other passages by the same and perhaps other authors;
- in the case of the difficult propositions of *Book I* of Apollonius's *Conics*, this reconstruction can provide the reader with a map for his orientation, moving from an enunciation to a demonstration and then to the final outcome as one moves from a plan to a section when studying a representation of architecture, and then reconnecting the drawings in the space of the mind and finally in reality.

It is well known that mathematicians have an innate difficulty in recognizing the role of graphic models in the elaboration of geometric thought, but there are also other authoritative opinions, and among all of them, Lucio Russo's is worth quoting in full: "Today we consider as independent three activities that were inseparably connected in Hellenistic mathematical practice: deductive reasoning, calculus and drawing" [Russo 2023, p. 59] [21]. And speaking of independence of activities and, I would add, of disciplines, it should also be noted how the scholastic habit of naming courses as being of 'analysis', of 'geometry', of 'drawing', and so on, has induced a separation that not only does not exist in reality, but is detrimental. If we look at the History of Representation, today we find Histories of Perspective, Drawing, Geometry and many other disciplines all separated from each other, whereas they are interdependent. On the other hand, I believe that Christian Wiener [Wiener 1884, pp. 5-61] was ahead of us and on the right track when, in 1884 he outlined a short history that moves with continuity from the perspective of the ancients to optics, topography, descriptive and projective geometry, pictorial and three-dimensional perspective and photogrammetry, and finally to the theory of chiaroscuro. A path that, as we know, certainly did not end with the advent of computer techniques.

### Notes

[1] As Girard Desargues said [Desargues 1639].

[2] The genesis of this famous theorem is narrated by Adolphe Quetelet [Quetelet 1867, pp. 144-147] and shows how tortuous the road to a result of such limpid simplicity can be. A result that, although being the result of the collaboration with Quetelet, was published by Germinal Pierre Dandelin in 1822 and earned him admission to the Belgian Academy of Sciences.

[3] The Italian edition of *Anschauliche Geometrie* (1932) was published in the *Universale scientifica Boringhieri* in 1972. Quetelet himself, noting the importance of this theorem, observes that the first to make use of it was Jean-Nicholas Pierre Hachette, in 1828 [Quetelet 1867, p. 145] (fig. 2), precisely in the second edition of his treatise on Descriptive Geometry [Hachette 1828, pp. 51-53], the first being published in 1822, and that Théodore Olivier, in 1847, had devoted a special study to it, which is included in the complements of his treatise [Olivier 1847, pp.V-VIII].

[4] He lived from 262 to 190 B.C. [Boyer 1980, p. 166], and Gino Loria writes of him, "So little is known of his life that up to now it has not been possible to decide whether or not he is to be identified with a contemporary astronomer of the same name. A later commentator describes him to us vain and bumptious, in strident contrast to Euclid, who was modest and always ready to acknowledge the merits of others. Of the improvements he suggested to the *Elements* of the great Alexandrian, we know little more than their existence; of his work on irrational quantities, a supplement to Euclid's *Book X*, we know only what an Arabic writer tells us about it; so of his work on the problem of constructing a circle touching three others situated in the same plane (a matter still designated by the name of 'The Problem of Apollonius') we only know the general

plan" [Loria 1930, p. 17]. The most important commentator among those mentioned by Loria is Pappus of Alexandria [Pappo 1560].

[5] Here I use the same adjective proposed by Gino Loria in his small volume on *Metodi matematici* (Mathematical Methods) [Loria 1919, pp. 77-83] to define this potentiality of geometric construction in general.

[6] On this subject there is an extensive multidisciplinary bibliography already cited in José Antonio Ruiz de la Rosa's essay [Ruiz de la Rosa 1987]. On the temple of Apollo at Didima, see Lothar Haselberger [Haselberger 1985]. The layout of the tympanum of the Pantheon has also been studied by Carlo Inglese [Inglese 2000, 2013].

[7] Apollonius' text is presented as an *ekphrasis* not unlike literary ones. Among the most famous examples of *ekphrases* there is Pliny the Younger's description of his Villa Laurentina, while writing to his friend Clusinius Gallus [Pliny 1973, pp. 314-329]. The reconstruction of the villa attracted the interest of many designers and scholars, from Vincenzo Scamozzi [Scamozzi 1615, pp. 265-268] to Karl Friedrich Schinkel (1833-1835) [*1781-1841. Schinkel*… 1982, pp. 158-161] to the competition announced in 1982 by the Institut Français d'Architecture [Porphyrios 1983, pp. 2-7]. A further and even more pertinent example is Leon Battista Alberti's *Descriptio Urbis Romae* [Alberti 2005], where the drawing is replaced by an alphanumeric code, just as in texts on geometry. The *Conics* originally comprised eight books, written in Greek. The first four have come down to us in the original language. The next three, from *Book V* to *Book VII*, have come down to us in an Arabic translation. The eighth book has been lost. The seven surviving books have all been translated into Latin, and almost all of these editions are illustrated. We recall the main editions, with particular reference to the first four Books. See Memmo 1537 [Apollonius Pergæi 1537]; Commandino 1566 [Apollonius Pergæi 1566]; Barrow 1675 [Apollonius Pergæi 1675]; Halley 1710 [Apollonius Pergæi 1710]. In addition, among the more recent editions: the critical edition by Heiberg [Apollonius Pergæi 1891; 1893]; the slightly abbreviated one by Thomas Little Heath [Apollonius of Perga 1896]; the excellent one by Paul Ver Eecke [Apollonius Pergæi 1923]; and, finally, the one by Robert Catesby Taliaferro and Micheal N. Fried [Apollonius of Perga 2013].

[8] To illustrate this paper, I elaborated the figures on the computer, as I have done for years now. But despite many attempts to alleviate the coldness of these drawings, I could not create images capable, in some way, of evoking the relationship between logic and drawing of which Lucio Russo speaks at length in his study of Greek scientific thought [Russo 2023]. In the end, I preferred to use a ruler and compass, and also having to distinguish some areas with colors, I imitated the graphics of Oliver Byrne [Euclid 1847] to whom homage must be paid for having translated into vivid images a luminous thought that is usually mortified by skeletal line drawings.

[9] "*Les propositions XI, XII et XIII, dont la lecture est assez ardue, sont les plus importantes du premier livre*": Paul Ver Eecke in Apollonius Pergæi 1923, p. XIII] ("The propositions 11, 12, and 13, whose reading is rather difficult, are the most important of the first book"). But already in the edition edited by Jesuit Father Claude Richard in 1655 we read, in one of the introductory chapters, entitled *Warning to the Reader who is a Scholar of Geometry*: "Section XIX – If and why the *Conics* of Apollonius are difficult. If Pappus of Alexandria, who excelled in the matters of geometry judged that, in order to understand the *Conics*, all his lemmas were necessary, in addition to the ninety by Apollonius himself, will they not be difficult?" [Apollonius Pergaei 1655, *Sectio XIX. An et cur difficilia sint Apollonij Conica*, without numbered pages, translated from Latin by the author].

[10] According to Apollonius's *Definition I*, the axis is the straight line passing through the vertex and the center of the circular base of the cone. Care should be taken not to confuse this segment with the straight line that belongs to two of the planes of symmetry of the cone and is perpendicular to the third, as in today's usage. The axis of Apollonius is, in general, distinct from the axis of symmetry: the two lines coincide only if the cone is straight. The term "axial triangle" comes from the original Greek "*ἄξονος τριγώνον*", which others translate [in Italian] as "triangolo per l'asse," from Johan Ludvig Heiberg's Latin edition "*triangulum per axem*". In the original text, in Greek, the definitions are not numbered; Heiberg distinguished them in Latin with the numbers 1 to 8. Here, for clarity, I have used Roman numerals.

[11] Note that, as reiterated in the caption for figure 4, the axial triangle does not, in general, coincide with the apparent contour of the cone itself with respect to the normal direction; this condition occurs only if the axial triangle belongs to a plane perpendicular to the base. Hence the non-projective nature of this image.

[12] In the case of the parabola, whose diameter has infinite length, the *latus transversum* is replaced by the distance between the vertex of the curve (defined in *Proposition 4*) and the vertex of the cone.

[13] It is surprising to note that there is no Italian translation of Apollonius of Pergas's *Conics*, a lacuna that is not so painful because of the language as because of the lack of a suitable iconographic apparatus. In fact, historical editions, as well as others, are all lacking in this respect. The translation into Italian of the reported passages is by the author of this paper.

[14] Every quadric cone, whatever directrix is used to generate it and, that is, a circle, ellipse, parabola or hyperbola, possesses two infinite arrays of circular sections. Apollonius is aware of this and constructs an opposite section in *Proposition 5*.

[15] The required condition of perpendicularity between the two intersecting lines of the base of the cone with the cutting planes, that of the triangle and that of the ellipse, might appear to be a constraint limiting the generality of the construction, but this is not the case, because what determines the shape and size of the ellipse is only the cutting plane, and therefore the axial triangle can be freely chosen.

[16] That is, the cutting plane and the base of the cone.

[17] That is, the parameter or *latus rectum*.

[18] Eutocius, in his commentary on *Proposition 11* of *Book I* [Apollonius Pergæi 1893, p. 217] explains how one can graphically represent the equation $BG^2 : (BA \times AG) = TZ : ZA$ which is analogous to the one we are interested in. And this concern of his is confirmed in the hypothesis that the drawing used by mathematicians of the time had operational significance. Adapting Eutocius's writing to the relations concerning the ellipse, namely to *Proposition 13*, we obtain the following reasoning (fig. 5): "Let $AK^2 : (KG \times KB) = ED : ET$ and what has been said is shown to be true, until proven otherwise. Let the rectangle ($KG \times KB$) in blue in the figure] be [drawn]. Let us apply to the side [*KG*] a rectangle of area equivalent to the square of side [*AK*] [in red] and let [*KS*] be the width of that rectangle [in yellow]."

Let us interrupt, for a moment, our reading of Eutocius to explain his argument in detail. In the scientific jargon of the time "apply to" stands for "build on"; therefore: let us build on that segment a rectangle whose height is *KG*, like the rectangle ($KG \times KB$), and which is equivalent to the square of side *AK*, that is, has the same area. This rectangle will, therefore, have *KG* as its height and a segment *KS*, as its base, the length of which must be calculated graphically. The two rectangles *BKG* and *SKG*, thus generated, have the same height and therefore their bases have the same relationship as their areas [Euclid 1970, VI, 1, p. 361]: $(KG \times KS) : (KG \times KB) = KS : KB$. Recalling, now, that $AK^2$ is equivalent to $(KG \times KS)$ by construction, we may write that: $AK^2 : (KG \times KB) = KS : KB$, and since $AK^2 : (KG \times KB) = ED : ET$ it follows that $KS : KB = ED : ET$. This relationship simply indicates that the segment *ET*, that is, the *latus rectum*, is to the diameter *ED* as *KB* is to *KS*. Therefore, Eutocius concludes in this way: "Let $KS : KB = ED : ET$ and thus we have obtained what we wanted and in fact, since $KS : KB = ED : ET$, it will be, on the other hand, [Euclid 1970, V, 7 coroll., p. 318] $SG : GB = ED : ET$, and it is, also, [Euclide 1970, VI, 1, p. 361] $KS : KB = SG : GB = AK^2 : (BK \times KG)$ [as was to be shown or demonstrated]".

[19] The Italian translation of the first thirteen propositions, annotated and illustrated as in this essay, is available at <https://www.migliari.it> (accessed 15 May 2024).

[20] When Apollonius has to indicate a rectangle he does not use four letters, but only the two he associates with the vertices of a diagonal.

[21] The pages that Russo devotes to drawing all deserve careful reading because they make clear what importance drawing had in the formation of Hellenistic geometric thought and how it assumed the value of an existential demonstration [Loria 1919, pp. 77-83].

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