The Masters of Vision. From Visionary Science to Visual Suggestions

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Abstract

The studies of Isaac Newton, in the 17th century, laid the foundations of classical physics. In the 19th century, however, some theories questioned Newtonian physics, whose weakness came from the application of concepts of Euclidean geometry to a space that may not be so. In 1817 Gauss, during his studies on the fifth postulate, formulated the hypothesis that for a point outside a line it was possible to draw more than one line parallel to it. Thus, he laid the premises of non-Euclidean geometry.
In 1884 Abbott published the novel Flatland*, in which he hypothesized a multi-dimensional space. The cultural debate thus opened up to visionary artistic expressions, derived from equally 'subversive' scientific concepts. Not to be neglected are also the studies of Poincaré that led to the topological space. These suggestions were anticipated by Möbius, in 1858, with the single-sided surfaces. The demolition of Newtonian dogmas also intertwined with perception studies. This led to the "impossible objects" of Reutersvär and Lionel and Roger Penrose. In the same years, also Escher shared the same passion for perceptual experiments.*
The paper aims to highlight the relationship between art and science which, between the 19th and 20th centuries, find a common 'visionary' inspiration. Often these paths are intertwined, sometimes one anticipates the other, but together they contribute to open pathways that mark the evolution of thought and art.

Keywords: Non-Euclidean geometries, Topology, Impossible objects, Möbius, Penrose, Escher.

The Euclidean dogma

Isaac Newton, in the 17th century, gave a decisive contribution to the foundations of classical physics. His studies hypothesized space and time as absolute entities. His statements were part of the undisputed dominance of the geometric principles expressed by Euclid in the *Elements*.
However, Euclidean geometry has an Achilles' heel. Although indirectly, the V postulate states that if two coplanar lines cut by a transversal form, on the same side, two angles whose sum is equal to a flat angle, will not meet and will be, therefore, parallel. This statement, however, does not have the qualities of 'demonstrability' and 'evidence' that, at that time, were necessary for it to be considered as a valid postulate. Euclid himself was aware of this, to the point that he did not use it for the demonstration of the first 28 propositions of the *Elements*. He only used it for one case. This awareness led him to consider the statement of the parallel lines as a theorem, although he never managed to find a valid demonstration. Following the failure of these attempts, he decided to reinsert it among the postulates [Agazzi, Palladino 1978, p. 48]. In the following centuries, there will be many attempts to exclude this proposition from the postulates, trying to demonstrate it as a theorem, but all will be unsuccessful.
Among the most ancient studies, we remember Proclus (5th century), who was firmly persuaded that "in the acquisition of geometric propositions, no weight should be

given to intuitive representations which are purely probable" [Proclus cited in Agazzi, Palladino 1978, p. 52]. His attempt failed when he introduced a hitherto unknown hypothesis: that the distance between two straight lines was finite. In fact, it was a new postulate that would make the demonstrative system collapse.
The attempt of Saccheri [1733], about XIII centuries later, will not obtain better results, but will be fruitful for future studies. Although unknowingly, he will pave the way for the birth of non-Euclidean geometry.
Saccheri proposed a demonstration *a contrariis* [1], based on 'absolute geometry' [2], which considered admissible two opposite hypotheses, implicitly excluded by Euclid: that for an external point to a straight line several parallels pass and, on the contrary, that none pass.
The demonstration failed because it was not able to demonstrate that the hypotheses admitted by absurdity were not true, but just this failure will determine its future success. "It became then clear –notes Sgrosso– that this proposition was to be considered effectively a postulate, assuming it together with the others, the Euclidean geometry was born, but assuming the excluded hypotheses, two different geometric theories were born, as valid as the first one" [Sgrosso 1986, p. 57]. These are the hypotheses on which some of the most enlightened scholars between the end of the 18th and the 19th century will work.

From the Euclid 'failure' to the 'visionary geometries'

Since he was a student, Gauss also tried to prove the V postulate of Euclid. In the beginning, he considered it as a theorem but soon he was convinced that it was indemonstrable and oriented his studies towards a system based on its negation. Starting from 1817 he worked on the hypothesis that assumes the existence of several lines passing through a point and parallel to an assigned line. He, with greater awareness, followed the path traced by Saccheri almost a century earlier. This opened the field to the hypothesis of a geometry very different from the one known until then, which Gauss at first called 'anti-Euclidean', then 'astral' and finally 'non-Euclidean'.
He never published the results of his studies. The scientific thought of his time was dominated by the figure of Kant, who considered Euclidean geometry as an inescapable necessity for thought. In the *Critique of Pure Reason*, published in 1781, the German philosopher defined space and time as *priori* forms [Kant 2000]. Thus, geometry was an absolute construction, based on indubitable principles [Mangione 1971, p. 182]. This cultural context decisively discouraged any position that questioned the Euclidean foundation of space. "I will not decide for a long time yet –wrote Gauss in one of his epistolaries– to elaborate for a publication of my very extensive researches on the topic, and this perhaps will never happen during my life, because I fear the shrieks of the Boeotians" [Agazzi, Palladino p. 75].
A few decades later, the studies of Hungarian Bolyai and Russian Lobačevskij will challenge the scientific community. They will propose concepts decidedly 'visionary' that, unbeknown to each other, will follow the analogous theories of Gauss. Bolyai and Lobačevskij demonstrated that for a point outside a line it is possible to draw several parallel ones to that given. This subversive hypothesis will open the field to a new geometry that Lobačevskij will call "imaginary" [3].
Riemann [4] moved in a similarly visionary direction. In 1851 Gauss put him on this path, assigning him the theme on which he would hold the dissertation for the achievement of the title *Privatdozent* [5]. Riemann also denied the Euclidean postulate but took an opposite route. He hypothesized an unlimited but not infinite space: "and it is precisely on the hypothesis of a finite space –says Andrea Giordano– that elliptic geometry was born, specifically highlighting the new idea of 'line', which here is precisely closed and finite. […] two lines (therefore all lines) of a plane meet, and consequently for a point of the plane no parallel to a given line passes" [De Rosa, Sgrosso, Giordano 2002, p. 218].
These studies will lead him to hypothesize the existence of a multidimensional reality. It is a further piece in the mosaic of the new non-Euclidean geometries that will be defined around the end of the 19th century. The Euclidean principles on which, for more than two millennia, the knowledge of reality were based are now definitely put in crisis by scientific concepts definitely visionary. This will push towards the search for new theoretical and scientific principles, able to support a new interpretation of reality.
The studies of Faraday and Maxwell on the propagation of electromagnetic waves will be decisive. These researches will definitely put the classical physics of Newton and his concepts of absolute space and time in crisis. The weak-

ness lies in its essential foundation: to apply concepts of Euclidean geometry to a space that could be not such. The time was finally ripe for a further leap that will radically change the conception of space.

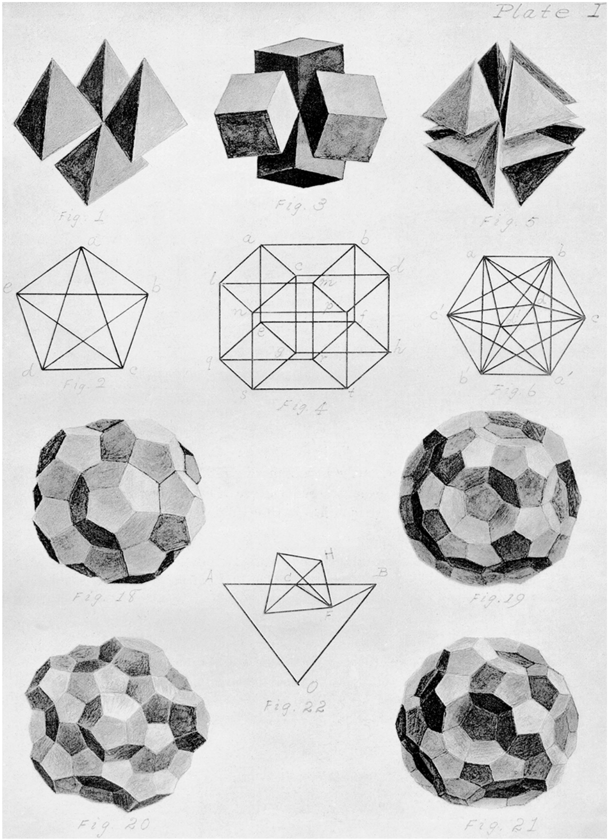
In 1905, Einstein published his theory of *Special Relativity*. He stated that space and time should be considered in a coordinated way. Time thus became a fourth variable, to be added to the three spatial dimensions adopted until then. Eleven years later he published a further development of this theory that he called *General Relativity* [Einstein 1916]. He hypothesized a four-dimensional space, in which the space-time entity (*Chronotope*) is curved by the presence of a mass and the gravitational field that it generates.

This will radically change the conception of space, pushing it towards a metageometric dimension. If in the presence of a gravitational field space-time is curved, then it can no longer be considered Euclidean. The theories on non-linear geometries of Gauss, Bolyai, Lobačevskij and Riemann are confirmed by the most advanced conceptions of physical space.

Therefore, Euclidean geometry is only one of the possible models of interpretation of reality. It is still valid for the world that can be experienced directly, but it was not the most suitable to support the new instances that were emerging in every field at the beginning of the 20th century.

Flatland

Abbott had already prepared the ground a few decades earlier. In 1882 he published what will become a classic of fantastic literature: *Flatland: A Romance of Many Dimensions*. He tells the story of a square, accustomed to living in a two-dimensional world, which discovers to it surprise that it belongs to a three-dimensional space (Spaceland). Its curiosity does not stop at this discovery but continues in visionary reflections, hypothesizing the existence of multidimensional spaces: "shall not, I say, the motion of a divine Cube result in a still more divine Organization with sixteen terminal points? […] And once there [in the four-dimensional space], shall we stay our upward course? In that blessed region of Four Dimensions, shall we linger on the threshold of the Fifth, and not enter therein? […] Then, yielding to our intellectual onset, the gates of the Sixth Dimension shall fly open;



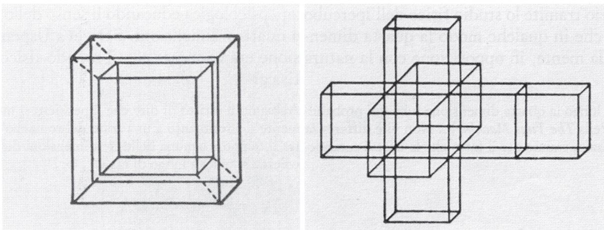


Fig. 1. W. I. Stringham, regular figures of four-dimensional space [Stringham 1880].

Fig. 2. H. P. Manning, representation of a hypercube, 1914.

Fig. 3. Left: T. van Doesburg, Une Nouvelle Dimension, 1925-1929. Middle and right: T. van Doesburg e C. van Eesteren, Maison particulière, 1924.

after that a Seventh, and then an Eighth" [Abbott 2004, pp. 68-69].
A few years later, the science fiction writer Hinton will publish an essay about the 4th dimension in which the term tesseract (hypercube) appears for the first time [Hinton 1888].
These were undoubtedly fascinating hypotheses but, at the time, many must have considered them bizarre. Actuality, these reflections were based on a maturing scientific debate and were filled with a stringent scientific logic. Multidimensional reality had no possibility of being perceived through experiential data but this did not exclude the possibility that it could be deduced by logical abstraction. As Emmer states: "the foundation of mathematics is in abstraction and therefore mathematics could appear far from physical reality" [Emmer 2003, p. 25]. Actually, logic and abstraction are sides of the same coin and contribute to the formulation of hypotheses and new scenarios that only later will be confirmed by scientific data.

The hypercube and metaphysical space

Between the pages of *Flatland*, albeit indirectly, there is the first description of a hypercube, a figure that will fascinate scholars and mathematicians but also inspire the art world. However, Abbott will not provide any illustration of such an entity. The first hypothetical representations of hypersolids are by mathematician Stringham who, in 1880, published an essay with a contribution to the definition of regular figures in four-dimensional space [Stringham 1880] (fig. 1).
A few decades later, the mathematician Manning [1914] published some graphic hypotheses of a hypercube: 'projections' from a four-dimensional space to a Euclidean one (fig. 2). Such representations were the result of a mathematical abstraction no less visionary than Abbott's literary descriptions. They captured the attention of artists and architects.
In number 5 of 1923 of *De Stijl*, eleven years after his death, the article by Poincaré *Pourquoi l'espace a trois dimensions?* was published. At the preface of the essay is the sentence: "The meaning of the fourth dimension for neoplasticism". It was a clear declaration of interest by the founders of the movement: Mondrian and Van Doesburg. The latter, in those years, clearly expressed a line of research in that direction (fig. 3). Describing a project for a private house in 1924, he wrote: "The new architecture is anti-cubic, in other words, its different spaces are not contained in a closed cube. On the contrary, the different cells of space (including balcony volumes, etc.) develop eccentrically, from the center to the border of the cube, so that the dimensions of height, depth, width and time receive a new plastic ex-

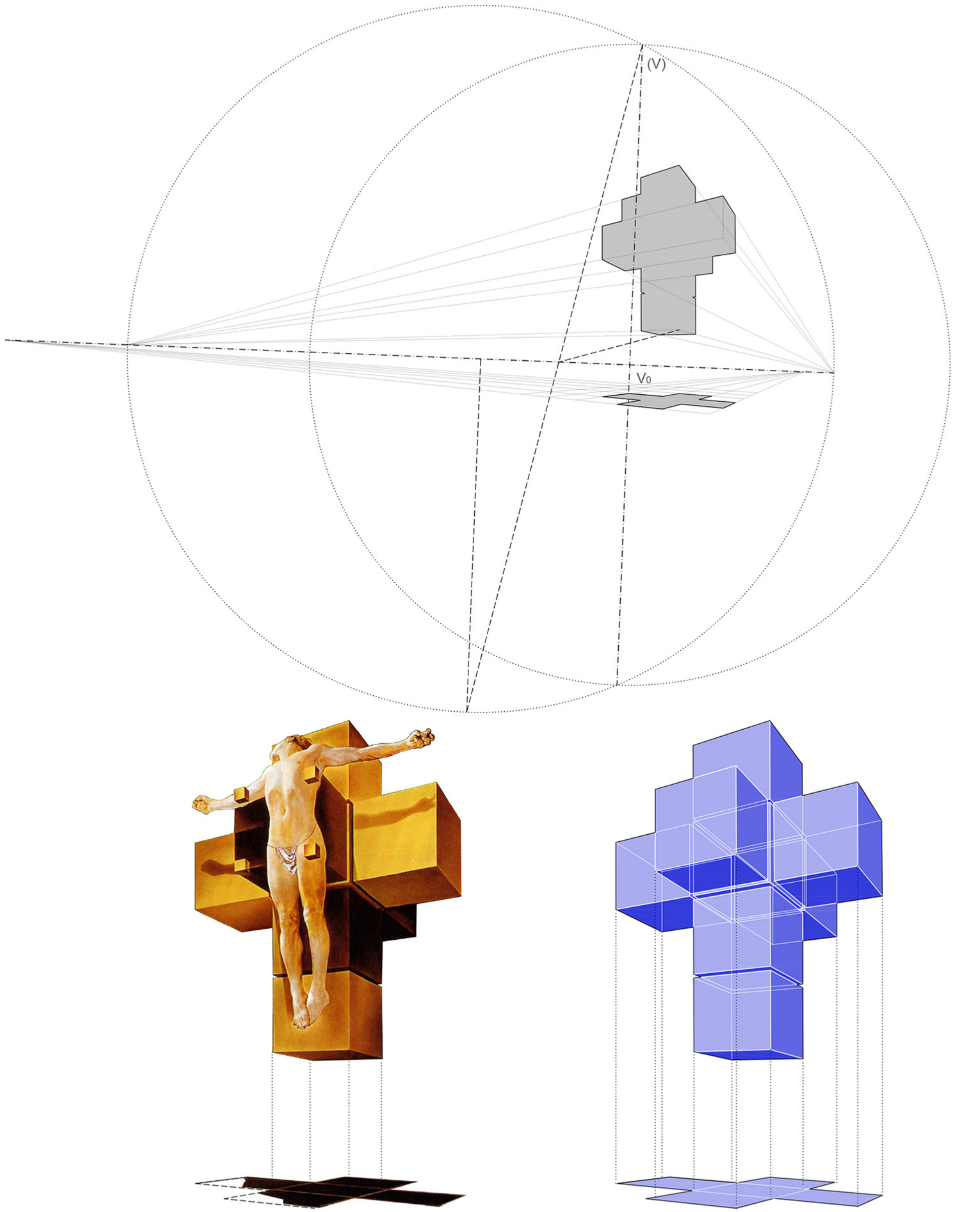
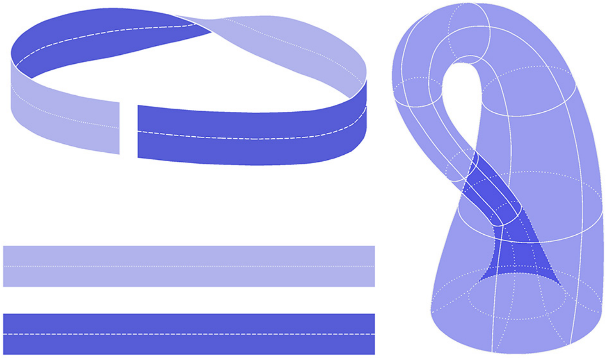


Fig. 4. S. Dalì, Crucifixion (Corpus Hypercubus), 1954. Top: analysis of the perspective structure. Bottom: details and redraw of the crucifix. Graphic elaboration by the author.



pression" [Van Doesburg cit. in Emmer 2003, p. 119]. Both the text and Van Doesburg's drawings referred to the representation of a hypercube even though, as Michele Emmer notes, he confused Flatland's four-dimensional objects –four-dimensional projections of Euclidean elements– with Einstein's space-time theory in which time constitutes a fourth dimension [Emmer 2003, p. 119].
The *tesseract* is an expression of scientific conceptions that subvert the usual empirical universe, but it also represents a link with unexplored universes that open up to fruitful artistic experimentation. In fact, the most visionary results will be expressed precisely in this field. Dalì explicitly showed his interest in the mixture of mystical and scientific dimensions found in four-dimensional space. In his work *Crucifixión* (*Corpus Hypercubus*) of 1954, he painted a Christ placed next to a cross suspended in the void, a clear representation of a hypercube (fig. 4). Everything happens without contact, in a metaphysical context dominated by an obscure natural landscape, in which the floor grid provides a weak anchorage to the empirical, logical and rational world. On it Dalì projects the hypercube, drawing a cross between the perspective grids. The new metageometric concepts, with their intrinsic need for abstraction and with a strong mystical and emotional character, constitute a bridge between the material world and the metaphysical dimension. Dalì's interests in the fourth dimension will continue in the following years. He came into contact with the mathematician Banchoff, keeping abreast of scientific developments regarding metageometric space. In 1979 he returned to the subject with the painting *In Search of the Fourth Dimension*. It is a surreal context in which citations of Raphael and Perugino are overlaid with symbolic elements, typical of the poetics of Dalì. In the foreground a dodecahedron is superimposed on the opening of what looks like a tomb: possible symbolic connection between reality and metaphysical space. In the background looms a 'soft clock', symbol of an eternal time that unifies and connects a visionary space steeped in Renaissance knowledge, Christian spirituality and pervaded by a disquieting mystery of oblivion.

Single-sided surfaces

Fig. 5. Single-sided surfaces. Left: Möbius ribbon, conformation scheme. Right: Kelin bottle. Graphic elaboration by the author.

Fig. 6. M. Bill, Endless ribbon, granite, 1953 (original version 1935). Baltimora Museum of Art.

De Stijl's posthumous interest in Poincaré testifies to the influence that the French mathematician's studies had on the artistic imagination of the 20th century.

In 1895 he published *Analysis Sitû*, the volume that will lay the bases of topological geometry. It "has as its object the study of geometric properties that persist even when shapes are subjected to such profound deformations that they lose all their metric and projective properties" [Courant, Robbins 1961, p. 353]. Such conceptions will open the field to very interesting visionary experiments.

A few years before Poincaré, in 1858, at the *Académie des sciences* in Paris, Möbius presented a long-neglected memoir on single-sided surfaces. In this work he described a shape with extraordinary expressive qualities: the 'Möbius strip' [6] (fig. 5). Almost eighty years were to pass before this geometric intuition found an application in modern art. In 1936, at the Milan Triennale, Max Bill presented the *Endless ribbon* (fig. 6). Unaware of Möbius' studies, he believed he had found a novel form. It was only later that he would discover the links with the geometric-mathematical studies of the previous century. The Swiss artist's interest in topology was not only linked to its aesthetic qualities but above all to the expressive-symbolic potential it offered. The analogy with the symbol of infinity triggers suggestions that go beyond mere shape. "If non-oriented topological structures existed only by virtue of their aesthetics, then, despite their exactness, I could not have been satisfied with them. I am convinced that the foundation of their effectiveness lies partly in their symbolic value. They are models for reflection and contemplation" [Bill 1977, pp. 23-25]. Rationality of mathematical thought and emotional expressiveness merge, generating unusual geometric configurations.

Vittorio Giorgini also moved in this direction. Between the 1960s and the 1970s he carried out some experimentation on single-sided surfaces [Mediati 2008, pp. 190-192]. His studies started from a critical point found in the conformation of 'Klein bottle' (fig. 5). In fact, it has a point of discontinuity in correspondence with the intersection which is determined when the tube penetrates the bottle. In order to solve this problem, Giorgini introduced a variation that eliminates the intersection and recovers the continuity between the internal and external surfaces (fig. 7). The result is extremely suggestive and elegant shapes, including the topological reinterpretation of the sphere and the torus, which in 2003 will be sculpted in alabaster by two artists from Volterra: Dainelli and Marzetti.

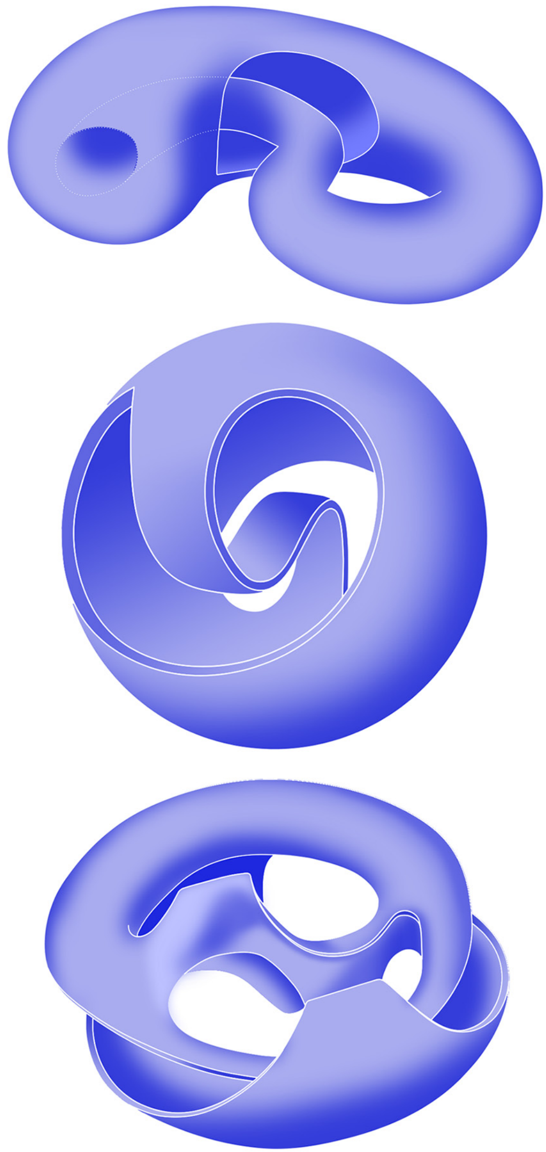


Fig. 7. V. Giorgini, Solids by Giorgini. From top: reinterpretation of Klein bottle; topological reinterpretation of the sphere; topological reinterpretation of the torus. Graphic elaboration by the author.

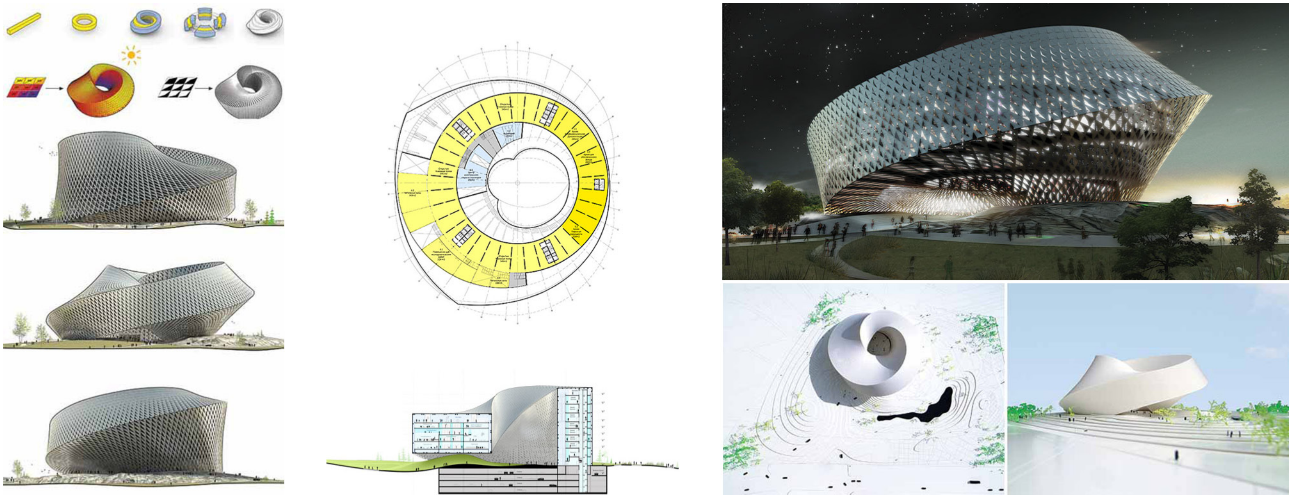


Fig. 8. B. Ingels Group, National Library, Astana (Kazakhstan), 2009. International competition winning project. The design is inspired by the Möbius strip.

These experiences are part of an approach that pushed Giorgini to the continuous search for organic forms. He was inspired by the studies of Thompson, a biologist and mathematician who believed that mathematical laws and physics played a crucial role in determining the forms and structures of living organisms.
Giorgini transferred these reflections to the field of architecture, rejecting the traditional techniques derived from 'classical geometry'. He privileged those 'techniques of nature' capable of configuring complex systems [Giorgini 2006, p. 34]. Therefore, Giorgini's free forms are the result of overcoming Euclidean space and of a hybridization between biological processes, mathematical laws and new expressive research.
The audacious shapes of topological space, from Möbius, to Klein, to Giorgini, up to the most visionary contemporary architectural designs, are the result of a reinterpretation of the concept of space and of an integration between art and science.
Computer graphics, techniques and new production processes allow, today, a reconnection between imagination and science, between theoretical and empirical space, elaborating forms that seemed unthinkable until a few decades ago. Scientific discoveries have radically changed the concept of space, giving it a topological dimension. Space is no longer a static cage, dominated by a rigid perspective structure, but it becomes fluid, changeable and malleable [Imperiale 2001].
It is in this context that contemporary 'soft' architectures come to life, as a result of a demolition of rigid Euclidean dogmas and that often take inspiration from the new visionary explorations in artistic and scientific fields (fig. 8).

Visionary perceptions and relative space

If empirical reality is only one of the possible realities, then the expressive potential of a visionary universe multiplies. Moreover, when the demolition of Euclidean and Newtonian dogmas is intertwined with an interest in perceptual studies, the field opens up to surprising impalpable and deceptive visions.
In reality, some experiments in the field of perceptual deceptions had been carried out since the 18th century by one of the most virtuous engravers. Piranesi, with the engravings of *Carceri d'invenzione* (Prisons of Invention), pushed the static Renaissance perspective to the extreme and opened the horizon to new interpretations of space. In the panel *Capriccio di scale, arcate e capriate* (1745-50) [7] he created a clever perspective artifice: two walls that

are parallel to each other are artificially connected by an arch that in turn appears parallel to the walls it connects (fig. 9). It is an obvious perceptual deception, anticipating the impossible objects that will be explored only in the following century and that will find great success in the second half of the twentieth century.
The studies of the Swiss crystallographer Necker are an example. In 1832 he drew a cube in which one of the posterior sides is superimposed on a front side. The result is a clearly unreal shape that can only exist in the 'illusory space' of the representation, a theme that will become recurrent in Escher's engravings (fig. 10).
A little over a century later, in 1934, Swedish artist Reutersvär also became interested in the theme. When he was only 18 years old, he drew an 'impossible triangle', composed of a series of cubes in axonometry that overlap in an apparently plausible manner but in obvious contrast to objective reality (fig. 11). Reutersvärd suffered from perceptual difficulties: dyslexia and difficulty in perceiving the size and distance of objects. These characteristics probably had a decisive influence on his experimentations and opened the field to visions that go beyond the Euclidean space. His research led him to create other unusual figures. In 1937, he drew the 'impossible stairs', well in advance of Escher and Penrose.
However, these experiments were only visionary intuitions without a wide following in the artistic and scientific fields. A decisive contribution to their success came only in 1958, when the British psychiatrist Lionel Penrose and his son Roger sent a short article to the *British Journal of Psychology*, which illustrated the 'Penrose stair' and 'triangle'. Two impossible objects that were inspired by Escher's experimentations, to which the essay referred [Penrose, Penrose 1958, pp. 31-33]. The paper, however, did not mention Reutersvärd's studies, which Roger discovered only in 1984.
In the same year that Lionel and Roger Penrose published their essay, Escher produced the engraving *Belvedere* (1958). In an apparently marginal position is a seated figure handling a 'Necker cube' and, at his feet, he has a sheet of paper with a scheme in which the crucial points of the deception are highlighted. Thus, Escher declares the geometric-perceptual inspiration used in the construction of the loggia that dominates the composition (fig. 10). As early as the 1940s, Escher had already created some engravings that reinterpreted the 'Möbius strip' and others that explored the potential of perceptual deceptions. Reality and space for Escher are expressed in a dimension of extreme

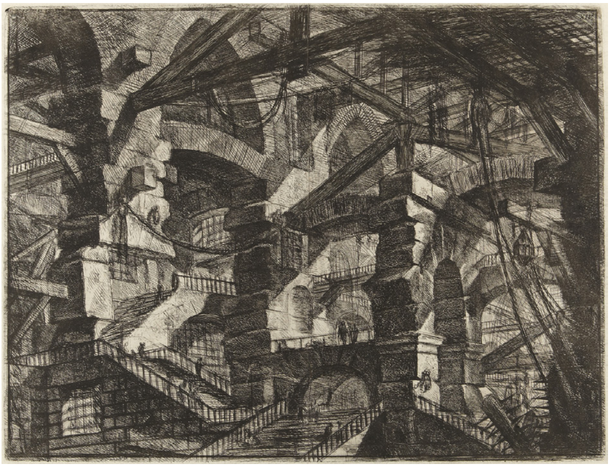


Fig. 9. G. B. Piranesi, Capriccio di scale, arcate e capriate. Taken from Carceri d'invenzione, 2nd edition, 1761, plate XIV, 415x548 mm.

'relativity' in which several worlds and perceptions intersect, imprisoning the protagonists in a universe in which there is no longer any distinction between horizontal and vertical, perception and reality, finite and infinite.
The two Penroses would send their essay to Escher from which he would draw further inspiration. The lithograph *Ascending and descending* (1960) is a reinterpretation of Penrose's staircase: two rows of hooded men traveling in opposite directions are imprisoned in an endless path (fig. 12). It is an evident perspective transposition of the infinite path of the 'Möbius strip'. In this engraving, topological concepts, perceptual deceptions, multidimensional spaces and perspective alterations intertwine, defining an unreal but perceptually plausible space, the result of a dreamlike and, at the same time, apparently rational vision. It is a subject that will be taken up again a year later with the engraving *Waterfall* (1961) in which a 'continuous bed' on which water flows replaces the steps. The perceptual deception imprisons the water in a perpetual flow, in which the force of gravity denies itself and produces an improbable endless circuit.
Escher's is a magical world, which finds its shape only in the privileged space of imagination and representation. Escher died in 1972 and did not have time to enjoy the many experimentations carried out on his works, with the help of computer graphics. The theme of impossible spaces and

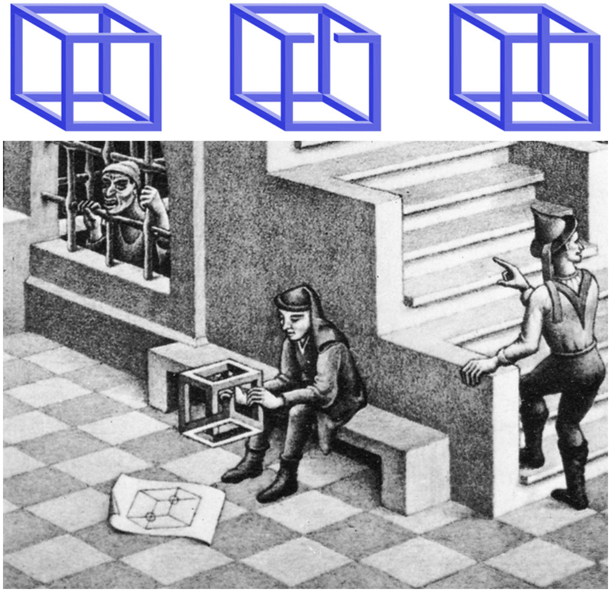


Fig. 10. Top: construction diagram of a Necker cube. Bottom: M. C. Escher, Belvedere, 1958. Lithograph, 461x295 mm. Detail.

Fig. 11. O. Reutersvärd, Impossible objects, 1934 et seq. Top: Stamps issued in 1982 by the Swedish government to celebrate Reutersvärd's work.

optical illusions has strong connections with the atopic space of the digital world. On the other hand, one of the first computer animations took place in the 1960s, in the Bell Laboratories of New Jersey, right on the 'Penrose stair'. In fact, the digital universe contains in itself all the ingredients of illusion: the possibility of creating unreal environments and simulating their concreteness using a mathematical and algorithmic structure. Once again science, mathematics and imagination collaborate, projecting the creative dimension towards new visionary expressions.

Conclusion

Art and science have a common matrix that drives the search for unexplored paths: a path that always moves the boundaries of knowledge further and further. Exploring unusual hypotheses, sometimes 'subversive', is the only way that produces innovation. Man's capacity for abstraction, that irrepressible instinct for 'vision', for overcoming the limits of appearance and the empirical world, are the foundation of all scientific and artistic evolution. Even in disciplines such as mathematics and physics, which appear to be firmly anchored in the experiential world, abstraction is the seed of every discovery: nothing can happen without imagination.
Between the 19^{th} and 20^{th} centuries, in a period of radical mutation, art and science find a common visionary ambition. The demolition of classical physics and Euclidean dogmas, the formulation of new multidimensional hypotheses, the theory of relativity, coexist with changes in the field of art. Perspective, which had dominated the world of representation since the Renaissance, is clearly challenged by the new artistic avant-garde. The demolition of the perspective universe, last anchorage to a Euclidean world, opens the field to visionary experimentations that, together with the new scientific instances define a new *Weltanschauung*.
A major contribution to the demolition of the old dogmas also comes from the use of the computer which, in recent decades, has facilitated the emergence of new formal research in both the field of art and architecture.
These paths are often intertwined, sometimes one anticipates the other, but together they contribute to open doors to intuitions, sometimes premonitory, that will mark the evolution of thought and art and will direct the perennial research of relationship between man and reality towards innovative and suggestive visions.

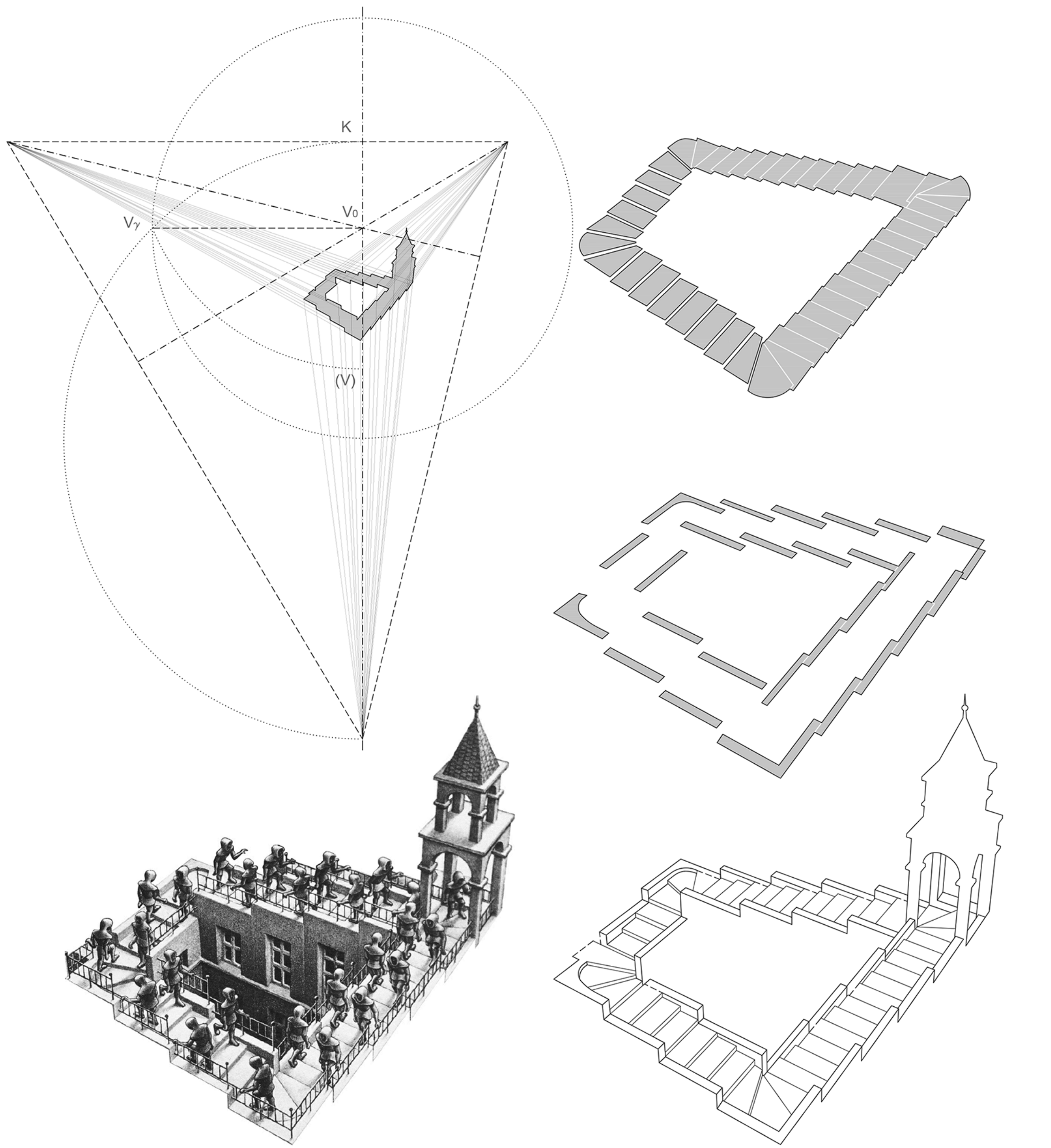


Fig. 12. M.C. Escher, Ascending and Descending, 1960, lithograph, 355x285 mm. Left: detail and analysis of the perspective structure. Right: graphic analysis with reference to Penrose stair. Graphic elaboration by the author.

Notes

[1] It is a procedure that allows to verify a proposition assuming as a starting point the negation of the same.

[2] 'Absolute geometry' derives from 'Euclidean geometry' but excluding the V postulate and all theorems derived from it.

[3] During a seminar held on February 11, 1826 at the University of Kazan, Lobačevskij made public his theories but the essay was never printed for fear of reactions from the scientific environment. Later he published some studies on "imaginary" geometry, the theory of parallel lines and a complete work [Lobačevskij 1856].

[4] He contributed to the foundation of 'Elliptic geometry'.

[5] The paper was published posthumously [Riemann 1868].

[6] Emmer finds this shape in some ancient references: in Roman mosaics of the 3rd century and in the harnesses for the horses of the troops of the Tsar of Russia in the 17th century [Emmer 2003, p. 68].

[7] The table appears with the numbering XII in the edition of 1745-50 and with the numbering XIV in the edition of 1761.

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