Catoptric Anamorphosis on Free-Form Reflective Surfaces

Francesco Di Paola, Pietro Pedone

Abstract

The study focuses on the definition of a geometric methodology for the use of catoptric anamorphosis in contemporary architecture. The particular projective phenomenon is illustrated, showing typological-geometric properties, responding to mechanisms of light reflection. It is pointed out that previous experience, over the centuries, employed the technique, relegating its realisation exclusively to reflecting devices realised by simple geometries, on a small scale and almost exclusively for convex mirrors. Wanting to extend the use of the projective phenomenon and experiment with the expressive potential on reflective surfaces of a complex geometric free-form nature, traditional geometric methods limit the design and prior control of the results, thus causing the desired effect to fail. Therefore, a generalisable methodological process of implementation is proposed, defined through the use of algorithmic-parametric procedures, for the determination of deformed images, describing possible subsequent developments.

Keywords: anamorphosis, science of representation, generative algorithm, free-form, design.

Introduction

The study investigates the theme of anamorphosis, a 17th century neologism, from the Greek ἀναμόρφωσις "riformazione", "reformation", derivation of ἀναμορφόω "to form again".
It is an original and curious geometric procedure through which it is possible to represent figures on surfaces, making their projections comprehensible only if observed from a particular point of view, chosen in advance by the author-designer.
In the application of the theoretical-practical fundamentals of the scientific method, the fortunate union between geometry, art and architecture is openly manifested, and is presented in design approaches and scientific and empirical research, provoking amazement in the observer.

The resulting applications require mastery in the use of the various techniques of the Science of Representation aimed at the formulation of the rule for the "deformation" and "regeneration" of represented images [Di Paola, Inzerillo, Santagati 2016].
There is a particular form of expression, in art and in everyday life, of anamorphic optical illusions usually referred to as " catoptric " or " specular ".
The in-depth study presented here focuses on this type of anamorphosis, which, as is well known, requires the additional presence of reflective devices that allow the deformed image to be deciphered.
The geometric phenomenon of anamorphosis has been known since the 14th century and finds its place in the

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broadest treatises on perspective [Accolti 1625; Baltrušaitis 1969; Gardner 1975]. Applications of this technique can be found in the works of painters such as Piero della Francesca and Hans Holbein the Younger [Di Paola et al. 2015; De Rosa et al. 2012].
The discipline was perfected in its scientific rigour in Europe, between the 15th and 17th century, as a result of the progress made in the fields of projective geometry and optics.
In recent times, the same principles have been applied in the realisation of temporary and permanent installations, on a multitude of supports and at different scales. Among the artists in this sense are Leon Keer, François Abelanet and Felice Varini.
Over time, interest in the technique grew because it was profoundly connected to the theme of illusion, paradox, oxymoron and the deception of ambiguous duplication of projection and, above all, with the "radical" metaphor that recognises the visual experience, and not only the artistic one, as essentially "spectatorial" in nature [Ugo 2002, p. 89].
When writing treatises, entire chapters describe the geometric genesis of such "illusions".
One of the most interesting protagonists of this complex conjuncture is the French mathematician and theologian Jean-François Nicéron, who entered the order of the Minims of St. Francis of Paola at an early age and devoted himself just as early to the study of optics and perspective [Nicéron 1638].
Experiments to date have focused on small-scale applications consisting exclusively of reflecting elements of a simple geometric nature (regular, ruled surfaces: straight cones and cylinders, planes or spheres).
This limitation of the devices used at the time is justified mainly by a number of factors. Firstly, the construction of a catoptric anamorphosis, using a free-form surface or one of a complex nature, generates a distorted image that can be several times larger than the apparent image, depending on the curvature of the surface.
Furthermore, this type of device necessarily requires the observer to be in a position opposite and above the surface or surfaces on which the distorted image is rendered.
Finally, a virtual image that is too large would be difficult to appreciate, thus failing to achieve the desired optical-perceptual effect.
The theory and practice of catoptric anamorphosis could offer numerous ideas for interdisciplinary research in the field of geometry applied to architecture, the figurative arts, visual perception and design.
Today, in fact, in contemporary architectural applications, the use of complex reflective forms using free-form surfaces with variable curvature is becoming increasingly widespread.
There are variable scale interventions already installed that could lend themselves to applications that exploit the anamorphic projective procedure, highlighting its potential applications, both from an expressive point of view and from a design point of view.
The use of this technique makes it possible to compensate for the aberrations typically caused by the curvature of reflecting elements, obtaining intelligible virtual images usually obtained with flat mirrors.
Using this expedient, the building envelopes could be more integrated in the surrounding context, avoiding the alienation of the user and ensuring the possibility of recognising non-aberrated shapes on a curved reflecting surface.
On the basis of these considerations, it is of interest to investigate and experiment new strategies that extend the field of application to free-form surfaces with the use of innovative digital representation tools.
Therefore, the essay introduces the projective method of the catoptric anamorphosis, through geometric-descriptive schemes, highlighting the theoretical principles and the peculiar characteristics of the optical-perceptual phenomenon.
The proposed methodology is introduced and defined, with the objective of extending and implementing the geometric method of the catoptric anamorphosis with digital tools and algorithmic-parametric control of the design process, which generalises the application to architectural elements of significant extension and of any geometric nature composed of reflecting surfaces.
In conclusion, in order to better explain the value and the perceptive impact of the observer and to validate the methodological path started, the discussion continues by presenting multi-scale application cases.

Catoptric anamorphosis: state of the art

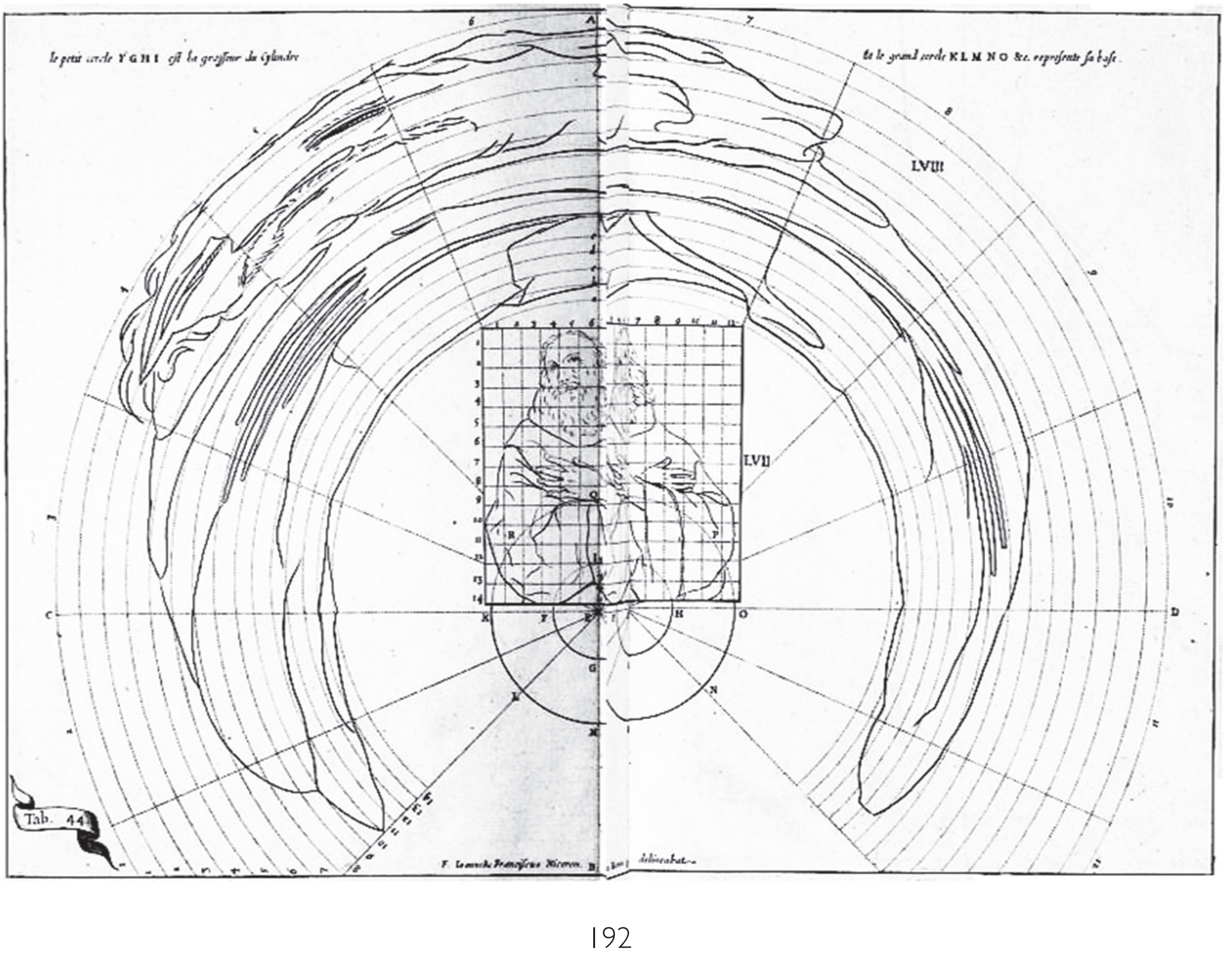
An image projected from its own specific projection centre onto one or more flat or free-form reflecting surfaces or generated by reflection in a deforming mirror (e.g. cylindrical, conical or pyramidal in shape) is perceived, by an observer standing at that viewpoint, without deformation. The optical aberration of the reflection compensates for the deformation of the anamorphic design and makes it proportionate and recognisable [Di Lazzaro, Murra 2019, p. 16].

The artifice of optical illusion is understood with marvel by the spectator as soon as they assume any observation position, at which point they perceive the represented figure as somewhat distorted and incomprehensible.
Graphic constructions are based on the laws of reflection and a peculiarity of this type of anamorphosis is the possibility of observing both a 'distorted' graphic and the 'correct' reflected image from the same point of view.
The introduction of the mirror as reflection, inversion and doubling makes the interplay of actor/spectator, reality/representation even more exasperated and intense. Vittorio Ugo effectively describes the typical nature of the phenomenon of observing "from the outside": "the only real artifice that makes it possible to represent in a single context the seen, the seeing and the vision is the referral operated by the mirror: a surface reflecting the real in a virtual image" [Ugo 2002, p. 88].
The use of devices made with curved mirrors, which historically have consisted mostly of ruled surfaces, such as right circular cylinders and right cones, polyhedral elements or, less frequently, spherical surfaces, has been documented since the 16th century. One of the precursors to deal with catoptric anamorphosis was J.L. Vaulezard in 1630, introducing empirical and illustrative experiments using cylindrical and conical reflecting surfaces. In the middle of the 17th century the Jesuit fathers, in the person of Gaspard Schott and his teacher Athanasius Kircher, dealt with the subject, laying the foundations for the spread of the phenomenon of perspective [Schott 1657]. In the 18th century, the theme became an essential practice to be discussed in treatises on perspective; an authoritative example is Ferdinando Galli Bibiena with his treatise: *L'architettura civile preparata su la geometria e ridotta alle prospettive. Practical considerations*, 1711 [Càndito 2010, pp. 71, 72].
More recent experiences are moving towards large-scale applications of catoptric anamorphosis [Čučaković, Paunović 2015] and towards the definition of non-two-dimensional deformed images, through the use of CAD programs, simulations with physical rendering engines and ad-hoc created codes [De Comité 2010; 2011; De Comité, Grisoni 2015].

Catoptric anamorphosis: geometric-descriptive schemes

The definition of distorted images was initially done by eye, and it was only after the first experiments that sufficient geometric knowledge was attained to allow their rigorous construction.
The catoptric anamorphosis, to allow the interpretation of the distorted image, requires the correct positioning of the reflecting device as well as the position of the observer.
The additional complexity due to the introduction of a reflecting device entails, therefore, the need to understand its geometrical characteristics thoroughly, also in relation to the mechanisms of light reflection and the rules of catoptrics, which can be schematised by means of linear rays.
Such information was acquired by treatise writers from the earliest evidence of the use of this type of device: however, only reflecting surfaces with rather simple geometries and almost exclusively for convex mirrors were described, while the few texts on concave mirrors generally resorted to simplifying hypotheses.
In 1638, the aforementioned Jean-François Nicéron published his treatise *La Perspective Curieuse*, in four books, devoted entirely to anamorphosis. In the third book, he delves into the subject of catoptric anamorphosis, describing the phenomenon through graphic examples.
The construction of the catoptric anamorphosis, known in the literature, involves drawing a reference anamorphic grid, on which the distorted image is then defined by hand. By way of example, we will focus in particular on analysing the construction of reflective anamorphic perspective projection on a convex cylindrical ruled mirror surface with a vertical axis.
In the projective space, in most cases, the projection setting, which generates the anamorphic image, must use an auxiliary plane perpendicular to the visual ray conducted by the point of view and projection centre V, on which the figure in true form is shown. Auxiliary planes not perpendicular to the direction of the visual ray would generate an image perceived in foreshortening, or would otherwise require additional artifices to compensate for this foreshortening.
Comparison with tables from treatises and drawings from the literature shows that various graphic devices are used to simplify the resolution process; these provide geometric approximations of the optical phenomenon which are compensated for by direct observation of the device with bi-ocular vision [Hunt, Harding MacKay 2011].
The exemplifications lie in the choice of the spatial position of the real grid and the distance of the observer from the reflecting surface (equivalent, in Nicéron's construction, to a projection from an infinite point of view) (fig. 1).

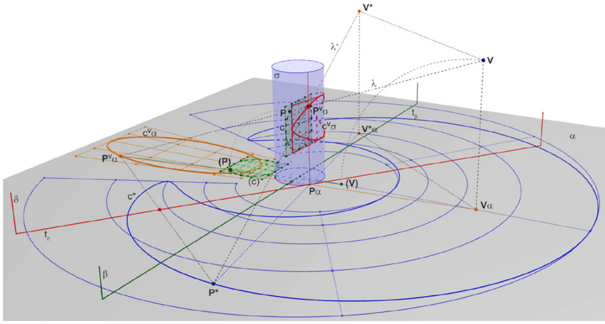
Fig. 1. J.F. Nicéron, orthogonal projection of a reflection anamorphosis on a cylindrical mirror surface of a human portrait (Nicéron 1638, Tab. 44, pp. 428, 429).

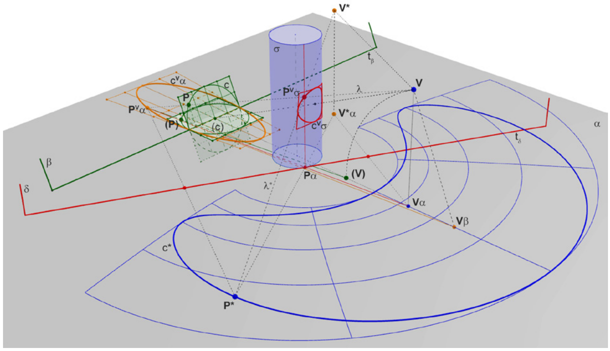


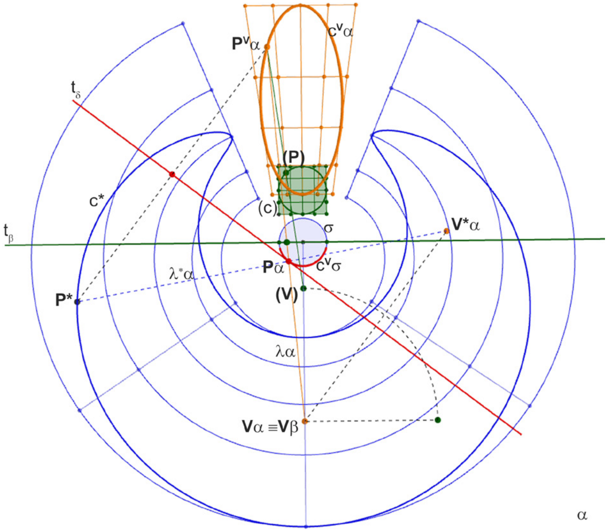
In the following we compare two representations of the anamorphic process made in two different configurations. From an angle "outside" the device, it is possible to observe: the centre of the projection *V*, the virtual figure *c*, a reference grid in true form placed on the auxiliary plane β, the anamorphic figure c^* on the horizontal plane α and its reflected image $c^V\sigma$ on the cylindrical reflecting surface σ with respect to the point of view *V*.
It should be noted that if the auxiliary plane β, to which the curve *c* belongs, were vertical and passing through the axis of the cylinder, the geometric construction would be easier to solve graphically, but the result would be less rigorous. In this way, the reference grid in true form, constituted by straight lines perpendicular to each other, would be deformed with approximation in a grid constituted by segments and, in the generality of the cases, by arcs of ellipses that can be easily traced (fig. 2).
If, on the other hand, the auxiliary plane β lies inclined with respect to the axis of the cylinder and orthogonal to the visual radius λ, the resulting network of the real grid would consist entirely of curvilinear profiles (fig. 3).
For both cases, the anamorphic procedure is defined as follows. Given a right-axis cylinder of known height, with a circular directrix of assigned diameter and resting on a horizontal reference plane α, a point of view of the observer *V* is fixed.
Once the auxiliary plane β is defined, a circle in true form *c* inscribed in a reference grid is drawn on it. The grid is placed in such a way that a pair of sides is in the same direction as the line t_β, intersection line of the plane α with the auxiliary plane.
Having chosen a point *P* belonging to the curve in true form *c*, a visual ray λ is drawn joining *V* with *P*. The ray intersects the cylindrical reflecting surface σ, in view with respect to *V* at the reflected point $P^V\sigma$.
According to the laws of reflection, for $P^V\sigma$ the tangent plane of reflection δ to the cylinder is represented. The anamorphic image P^* of point *P* on the plane α is determined by means of the counter-observer V^*, placed in symmetry to the plane δ with respect to *V*.
Iterating the procedure and moving the point *P* on the curve in true form *c*, two loci are described: c^* which represents the anamorphic projection of the assigned curve and $c^V\sigma$, the non-planar curve belonging to the reflecting cylindrical surface.
In order to obtain the image of the original drawing, i.e. the circumference and the circumscribed grid free of de-

Fig. 2. Perspective view of the catoptric anamorphosis procedure with a cylindrical reflecting surface. Virtual figure c belonging to the vertical auxiliary plane β (authors' drawing).

Fig. 3 Perspective view of the catoptric anamorphosis procedure with a cylindrical reflecting surface. Virtual figure c belonging to the oblique auxiliary plane β (authors' drawing).







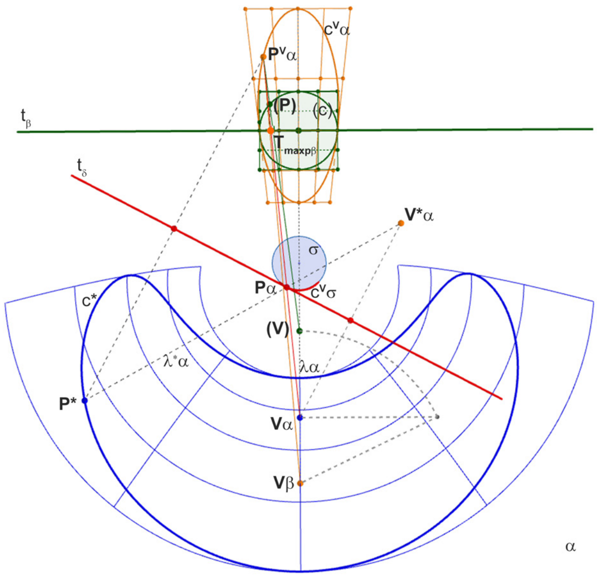


Fig. 4. Orthogonal projection of the catoptric anamorphosis procedure with a cylindrical reflecting surface and a vertical auxiliary plane β. Geometric construction with a ruler and a compass (authors' drawing).

Fig. 5. Orthogonal projection of the catoptric anamorphosis procedure with a cylindrical reflecting surface and a oblique auxiliary plane β. Geometric construction with a ruler and a compass (authors' drawing).

formations, it is necessary to observe the anamorphic curve reflected on the cylindrical mirror from the observation point *V*; by doing so, the circumference *c* will be superimposed on the non-planar curve $c^V\sigma$.
The comparison of the two approaches shows the different geometrical nature of the resulting anamorphic curves, according to the above-mentioned observations. This second approach, of general validity, which takes into account the rules of catoptrics, will be used for the algorithmic implementation described in the next paragraph.
In the proposed digital graphical constructions, the methods of representation of descriptive geometry are employed, applying traditional procedures of plane transformations of geometric entities with straightedge and compass (figs. 4, 5).
By overturning the auxiliary plane β on the plane α, around the intersection line t_β, a homology is established that puts the two superimposed planes in bi-univocal correspondence.
The point *(V)*, the overturning of the point of view *V*, the trace t_β, the overturning hinge of the auxiliary plane, and the point *(P)* belonging to the overturned curve *(c)* are identified.
We define, then, V_β as the projection of *V* on the plane α according to the direction of the line of maximum slope of the auxiliary plane β. We define $T_{maxp\beta}$ as the trace point of the line of maximum slope of the plane β on the plane α led by *(P)*. Finally, we determine $P^V\alpha$, as the intersection of the straight lines $V_\beta T_{maxp\beta}$ and *(V)(P)*.
A homology is then established with the following elements: homology centre *(V)*, homology axis t_β, and a pair of corresponding points *(P)* and $P^V\alpha$.
Given the fundamental properties, corresponding points in a homology are aligned with the centre and distinct corresponding lines secede on the axis. The point $P^V\alpha$ describes the ellipse locus $c^V\alpha$ at the variation of *(P)* on *(c)*.
Running a line through $V\alpha$ and $P^V\alpha$ we determine: the point of intersection $P\alpha$ belonging to the circular directrix of the cylindrical reflecting surface σ, the line t_δ, tangent to the directrix and, by axial symmetry to the latter,

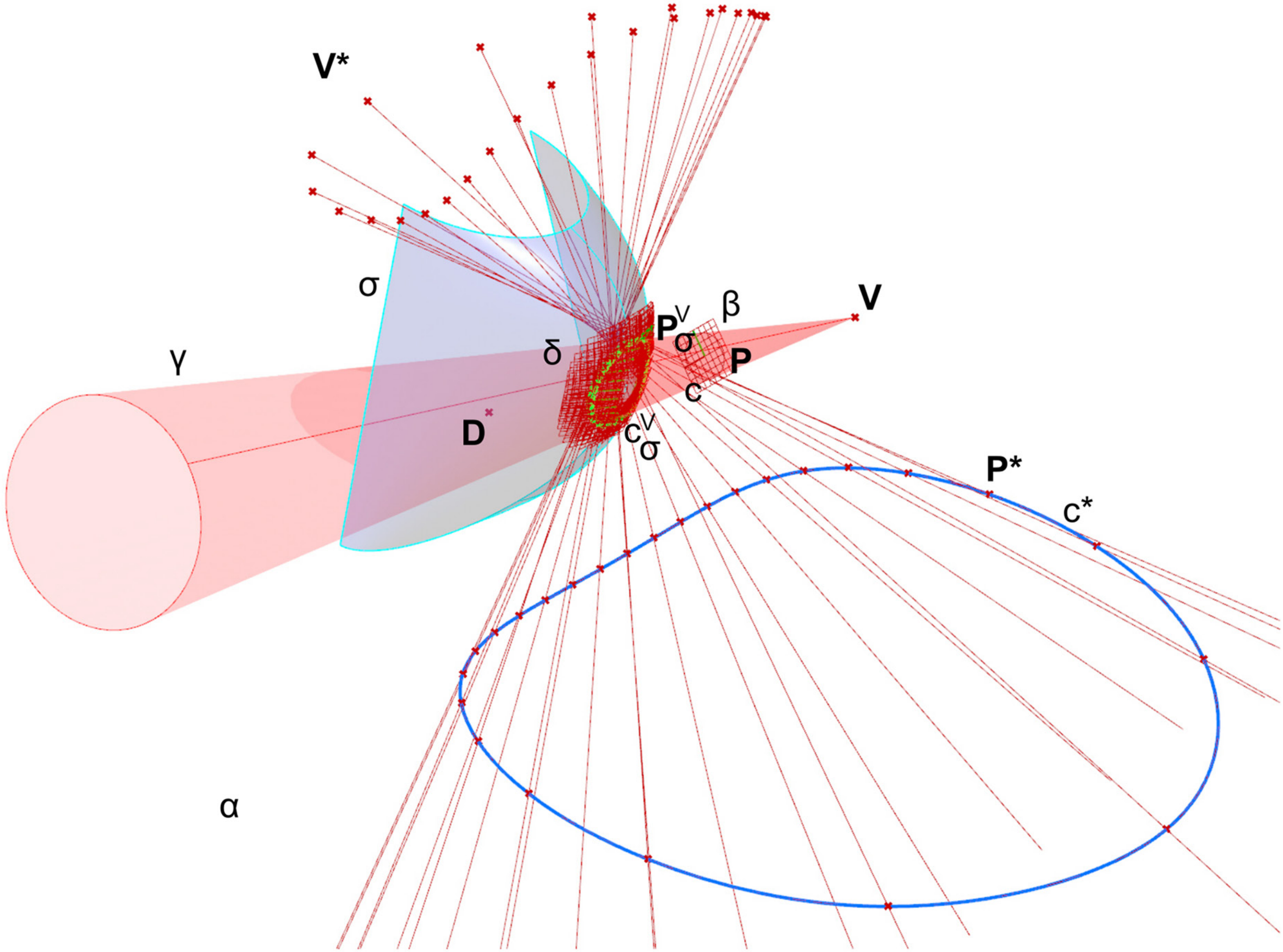


Fig. 6. Simplified scheme illustrating a conventional arrangement to reproduce the illusory process of catoptric anamorphosis on convex surfaces (authors' drawing).

the point P^*, symmetrical to $P^V\alpha$. The point P^* describes, when *(P)* moves along *(c)*, the locus c^*, the deformed anamorphic curve.

Catoptric anamorphosis on variable curvature free-form surfaces. Method, instruments and case studies

As anticipated, the methodology employed in this study assumes considerations related to the physical phenomenon of the reflection of a visual ray. In particular, the reflection mechanisms on a planar mirror and on curved mirrors with regular geometries (concave and convex spherical mirror, parabolic mirror) are considered known from the state of the art. By analogy, the reflection mechanism on cylindrical and conical mirrors, historically used in the realisation of catoptric anamorphosis, is considered known, as described in the previous paragraph.
This simple tracing is no longer possible regardless of the chosen viewpoint if the reflecting surface does not have known and easily defined geometrical properties. For example, a generic double-curved NURBS surface without symmetry planes will cause a nonequivalent controllable deformation of the elements of the reflected grid [Eigensatz et al. 2010; Flöry, Pottmann 2010; Wallner, Pottmann 2011].
Realisations of catoptric anamorphosis using complex and/or composite reflective surfaces are uncommon and usually the result of an empirical procedure in which the author –predominantly an artist– calibrates the desired result by successive approximations with little or no geometrical rigor.
An algorithmic point-by-point procedure is proposed to define the distorted image by loosening the constraints given by the geometric characteristics of the mirroring surface [Buratti 2012].
Using the phenomenon of determining the reflected image of a point from a planar reflecting surface by determining a counter-observer, the procedure is iterated for the points constituting the true-form curve, then interpolating the resulting anamorphic points.
The specular plane for each point is identified in the plane tangent to the reflecting surface at the virtual point lying on this surface, as outlined below.
The possibility of generalising the realisation of the catoptric anamorphosis makes it possible to extend the field of application beyond the small scale typical of this type of phenomenon, such as for architectural scale realisations.
In the context of the use of this algorithmic procedure by a designer, the following significant elements are identified: the point of view or observer *V*, the direction of the visual ray given by the destination point of view *D*, the curve in true form *c*, the reflecting surface σ, and the receiving support α.
In addition to these significant geometric elements, two further values are used: a homothetic scaling coefficient of the curve and the number of sampling points (resolution) (figs. 6-7).
Once the direction of the visual ray has been defined, the curve in true form *c* is positioned perpendicular to it on an auxiliary plane β representing the perceptual result of the anamorphic procedure. We define a generic cone γ having as directrix the curve in true form, as vertex the observer and as generators the visual rays. This cone γ intersects the surface σ defining the virtual curve $c^V\sigma$.
For each sampling point *P* on the σ surface, the mirroring plane δ tangent to the σ surface is then defined. Each δ plane is used in the determination of the counter-observer V^*, reflection of point *V* with respect to the plane, for each of the sampling points (in some particular cases, distinct sampling points have the same counter-observer, but this is not true in general and there is a biunivocal relationship between sampling points and counter-observer).
The line joining each counter-observer V^* with the corresponding sampling point *P* is identified.
Where this line meets the surface α, the anamorphic point P^* is determined.
The procedure thus described is repeated for each of the sampling points $P_1...P_n$.
Once the anamorphic points have been determined, the anamorphic curve is reconstructed by interpolation.
In order to highlight the educational, popular and artistic-expressive potential that the technique could offer in current contexts, some examples of applications with different quadric and free-form surfaces of different complexity are presented: a small-scale application, such as a mathematical machine for educational and museum use (fig. 8); a reflective geometry similar to the land-art piece 'Cloud Gate' by the artist Anish Kapoor [1] (fig. 9); a possible large-scale application using the curvilinear reflective surfaces of the Palazzo della Regione Lombardia in Milan by Pei Cobb Freed & Partners (fig. 10); and finally a model of the free-form reflective surface of the

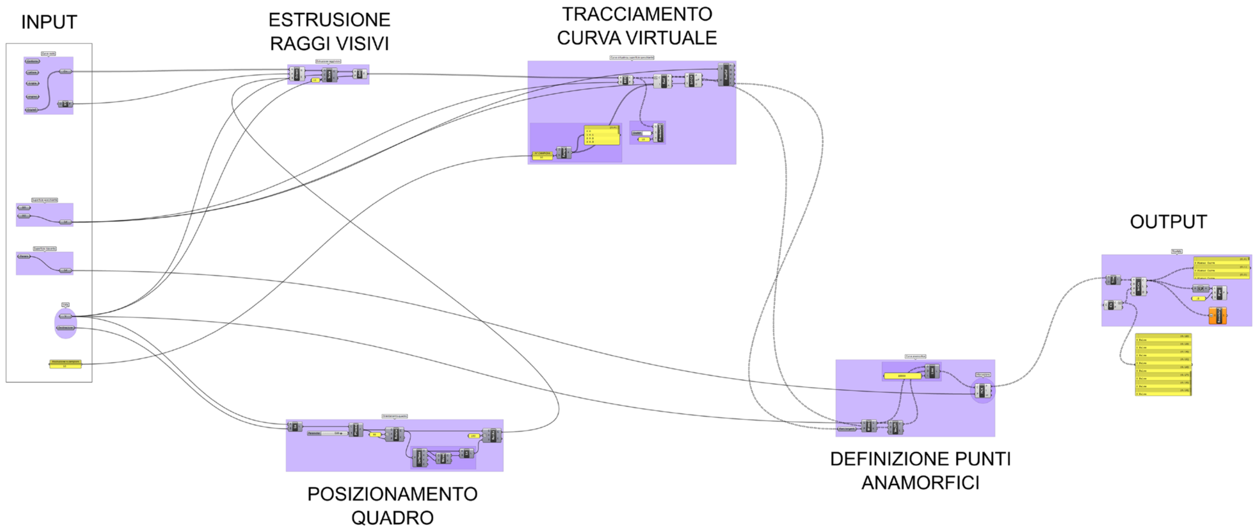


Fig. 7. Definition of the structure of the visual algorithm controlling the generation of the illusory process in the Grasshopper plugin workspace (authors' drawing).

Boijmans Van Beuningen Museum in Rotterdam by the studio MVRDV (fig. 11).
For each of these examples, a visualisation from the privileged viewpoint of the observer and isometric views representing the anamorphic curves in true form are shown.

Conclusions

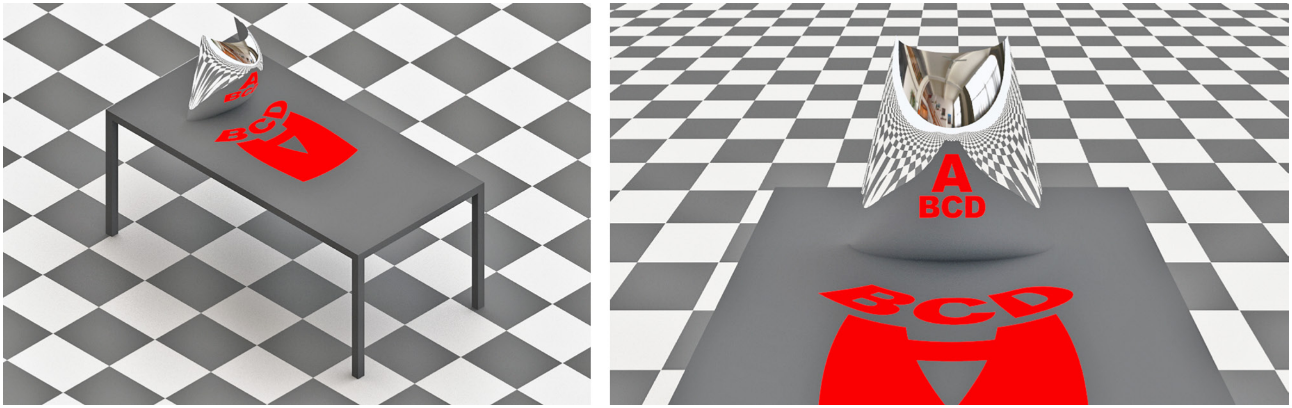
The proposed methodology offers the possibility of implementing geometric processes of catoptric anamorphosis of remarkable expressive potential in multi-scalar design contexts with reference to reflecting elements of relevant extension and geometric complexity with a generalizable parametric control [Rossi, Buratti 2017; Saggio 2007].
In particular, in the use of generic free-form surfaces, the defined method guides the designer in the construction of the anamorphosis, allowing rigorous measurement of the deformed figures and the corresponding reflected images on the reflecting surfaces, predicting, with reference to the chosen point of view, the outcomes of the projective phenomenon [Bianconi, Filippucci 2019].
The algorithm developed makes it possible to determine the catoptric anamorphosis for almost any surface or portion of a convex surface, but does not return an equally reliable result if the surface is concave or presents a change of concavity. This is due to the physical-optical characteristics of the light reflection phenomenon, which in the well-known examples of reflection in spherical concave and paraboloid mirrors presents, in the general case, an inversion of the reflected image.
In the case of a fully concave surface, the resulting image is more difficult to manage, while in the case of a change of concavity, the inversion of the reflected image makes it impossible to determine an unambiguous anamorphic image.
This is true for both free-form and ruled surfaces with these characteristics.

Notes

[1] <http://anishkapoor.com/> (accessed 2020, 10 November).

Fig. 8. Catoptric anamorphosis, example of small-scale application, mathematical machine of teaching laboratory; on the right, rendering of the privileged viewpoint (authors' drawing).

Fig. 9. Catoptric anamorphosis, example of medium-scale application, the inscription "Diségno 2020" reflected on a geometry similar to the land art work "Cloud Gate" by the artist Anish Kapoor; on the right, rendering of the privileged point of view (authors' images).



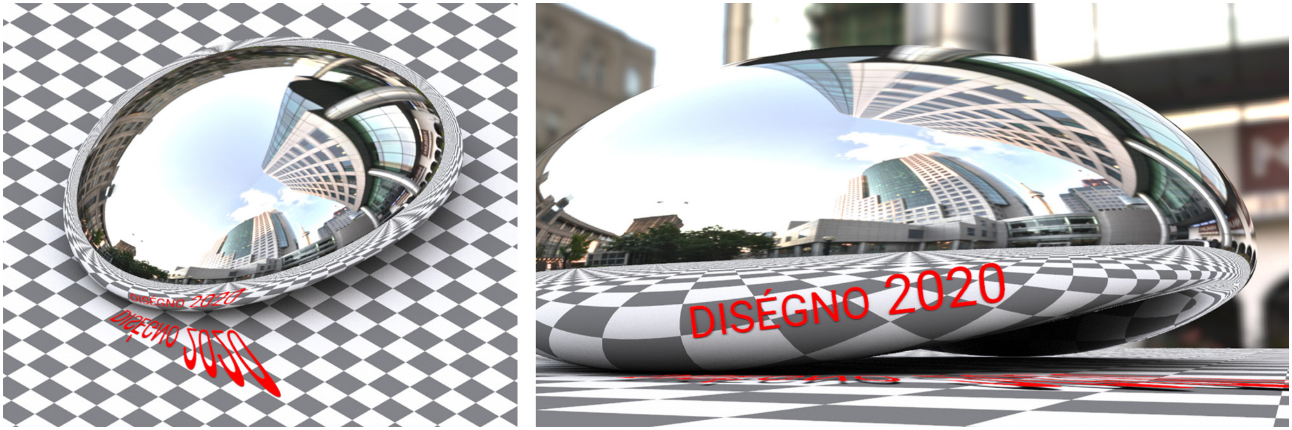
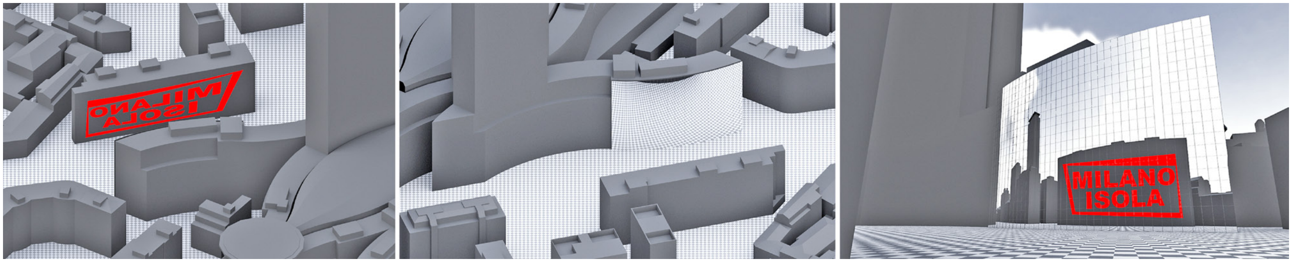
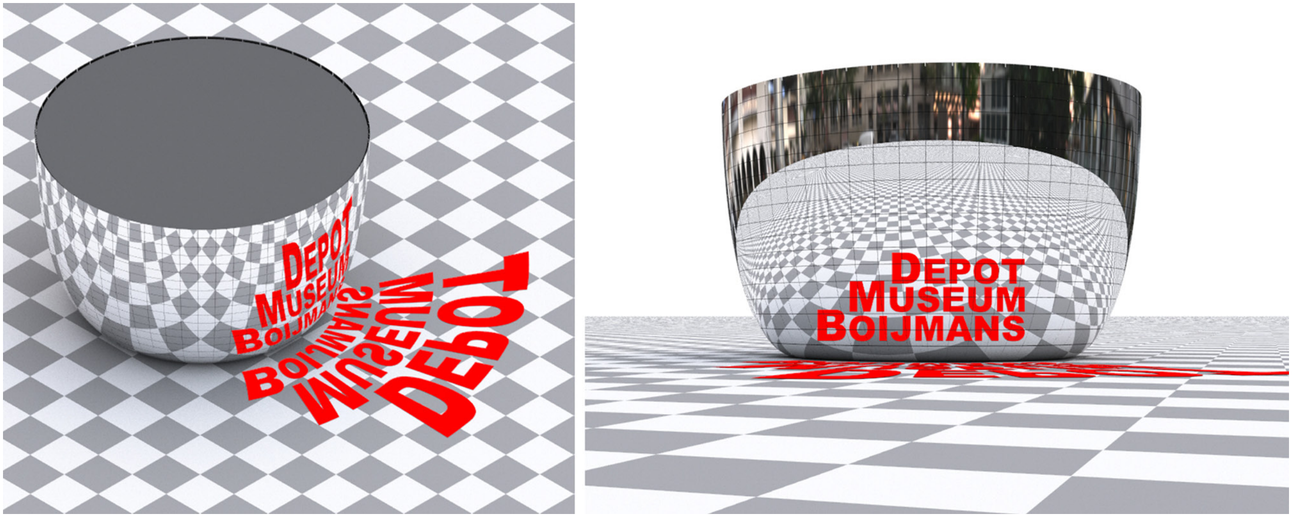


Fig. 10. Catoptric anamorphosis, example of a large-scale convex surface. Render reproducing the reflection on a building elevation; on the right, privileged viewpoint. Reproduction of the shape of the main elevation of the Palazzo Regionale Lombardia, Milan. (image by the authors).

Fig. 11. Catoptric anamorphosis, example of a large-scale complex surface. Render reproducing the shape of the Boijmans Van Beuningen Museum, Rotterdam; on the right, privileged viewpoint (authors' image).





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