Theories and Methods for Development of Developable Ruled Surfaces and Approximate Flattening of Non-developable Surfaces

Mara Capone

Abstract

Geometric genesis of surfaces and knowledge of their properties are basis for solving many problems, both constructive and measurement. A developable surface can be manufactured starting from a flat "strip", using a flexible and non-deformable material. This is a very important feature of the surface. Geometry studies the properties that don't change and, therefore, the shape of the "strip" to obtain a certain configuration, after a series of rigid movements. The paper addresses different methods to define the development of developable surfaces and non-developable surface "flattening", or approximate development. The aim is to study the relationships between methods, illustrated in some treatises, and the applications that can derive from the use of parametric tools. We are going to create an overview of different approaches, that have defined the bases of differential calculus for the study of ruled surfaces properties, and of different methodologies that allow to determine their development. Starting from the first definition of surface by Aristotle in De Anima *(384-322 BC) and the ambiguous definitions by Amédée François Frézier (1682-1773), we analyzed the studies of Leonhard Euler (1707-1783) and Monge's main work on developable surfaces (1795).*

Keywords: Developable surfaces, Ruled surfaces, Double curvature surface, Parametric modeling.

Introduction

Geometric genesis of surfaces and knowledge of their properties is the basis for solving many problems, both constructive and measurement. A developable surface can be manufactured starting from a flat "strip", using a flexible and non-deformable material. Developability is a very important feature of a surface. Geometry studies the properties that don't change and, therefore, the shape of the "strip" to obtain a certain configuration, after a series of rigid movements, without stretching or tearing.

Theories

Differential classification of surfaces introduced by Leonhard Euler (1707-1783), and subsequently used by Monge, allows us to group surfaces according to the definition of curvature, which will be precisely defined by Carl Friedrich Gauss in 1902 [Gauss 1902, pp. 10-20], in four categories: surfaces with zero curvature, surfaces with positive curvature, surfaces with negative curvature, surfaces with variable curvature.
The curvature of a curve in *P* is *k*, where $k=1/r$ and *r* is the radius of the osculating circle of the curve, we can define as the main sections of a surface, the sections of the surface obtained with planes passing through the normal to the surface in *P*, with minimum and maximum curvature.

Superfici sviluppabili
Curvatura nulla k=0
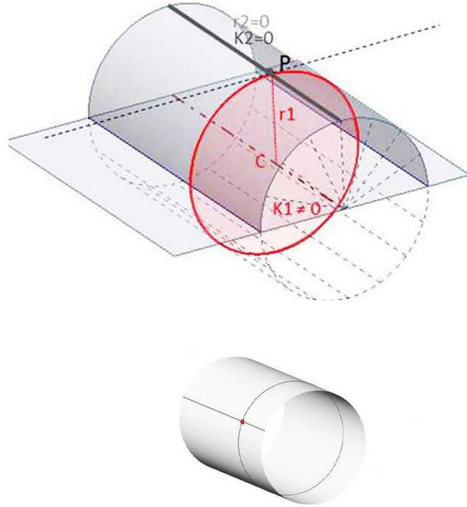
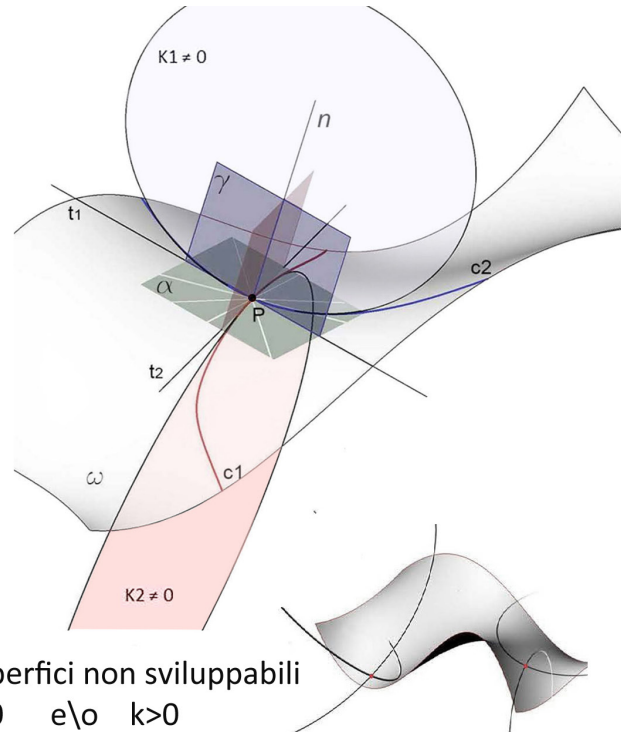Superfici non sviluppabili
k<0   e\o   k>0

Fig. 1. Osculator circles and Gaussian curvature of a surface.

Leonhard Euler shows that the main sections of a surface belong to orthogonal planes.

Gaussian curvature is the product of the two main curvatures, so it can be positive, negative or equal to zero: it is positive when the osculating circles of the main sections are on the same side of the tangent plane, negative when they are on opposite sides, zero when one of the two main sections is a straight line.

The surfaces with zero curvature are some specific ruled surfaces, also called *developable* (fig. 1).

The Jean Pierre Nicholas Hachette's book is very important to study ruled surfaces. He classifies these surfaces into two categories: the developable surfaces, which are obviously ruled, and the ruled surface, which for the French scholar, are the non-developable ruled surfaces [Hachette 1828] [1].

On the other hand, Euler and Monge are the first to study systematically the ruled surfaces properties according to the principles of "differential geometry" [Snežana 2011, pp. 701-714]. Both propose a generalization of the question, even if they never explicitly refer to the concepts on which this classification is based, and never speak about an osculating circle or an osculating plane [2].

Euler explicitly poses the problem of surfaces development. In *De solidis quorum superficiem in planum explicare licet* he defines the geometric conditions of a surface so that it can be developed: "*Notissima est proprietas cylindri et coni, qua eorum superficiem in planum explicare licet atquea deo haec proprietas ad omnia corpora cylindrical et conica extenditur, quorum bases figuram habeant quamcunque; contra vero*
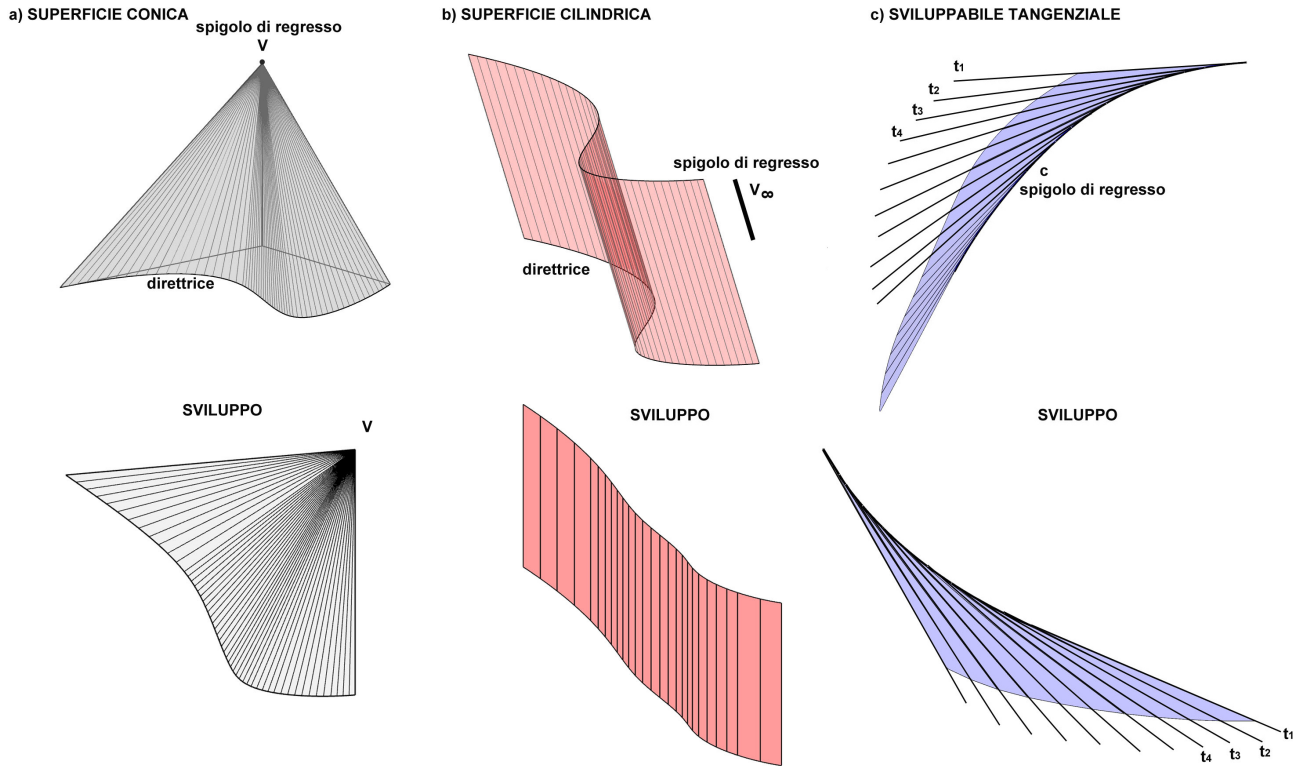
a) SUPERFICIE CONICA

spigolo di regresso
V

direttrice

b) SUPERFICIE CILINDRICA

spigolo di regresso
V_∞

direttrice

c) SVILUPPABILE TANGENZIALE

t_1
t_2
t_3
t_4

c
spigolo di regresso

SVILUPPO

V

SVILUPPO

SVILUPPO

t_1
t_4 t_3 t_2

Fig. 2. Developable surfaces classification according to edge of regression. Determination of their developments.
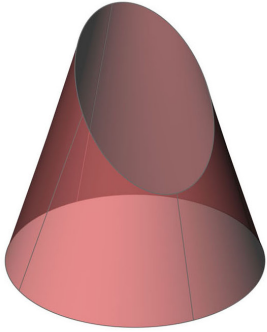
Starting from the research of the conditions that make a
surface developable, Euler's main merit is to have clearly
related the principles of analytic geometry and differen-
tial geometry.

Monge, who introduces a new family of developable sur-
faces, opened up questions that still today are the basis
of the different approaches for construction of complex
shapes [4]. However, who has introduced a new family
of developable surfaces is Monge, dealing with questions
that still today are the basis of the different approaches
for the complex shapes fabrication. In fact, using the theo-
rem that Monge illustrates in his lectures on Descriptive
Geometry [Monge 1798] to demonstrate the domain of
existence of a generic ruled surface, he defines a particu-
lar surface generated by a line that moves along a curve,
tangent to the curve: this surface is called tangential de-
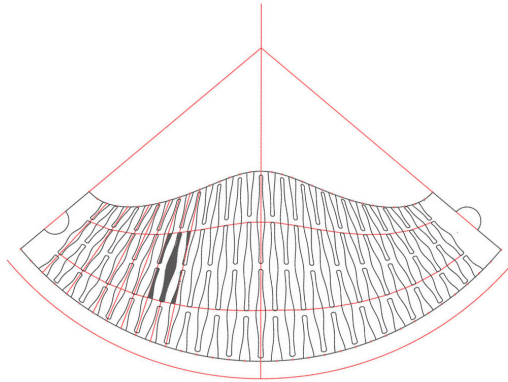velopable [Monge 1795, p. 130] [5].
Therefore, based on the studies of Monge, Euler and Ha-
chette, we arrive at a general definition of developable

Fig. 3. You can easily manufacture conical and cylindrical surfaces using a "flexible" but not "deformable" material (graphic elaboration by the author).

1. Superficie conica - sviluppabile   2. Sviluppo   3. Costruzione
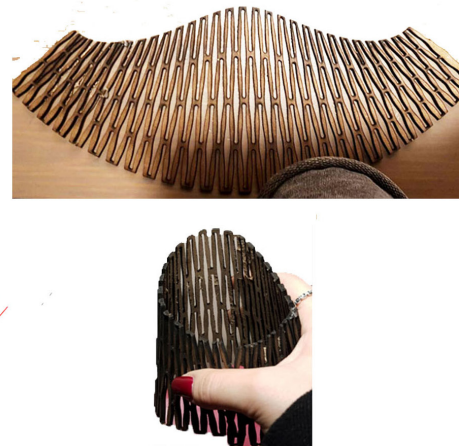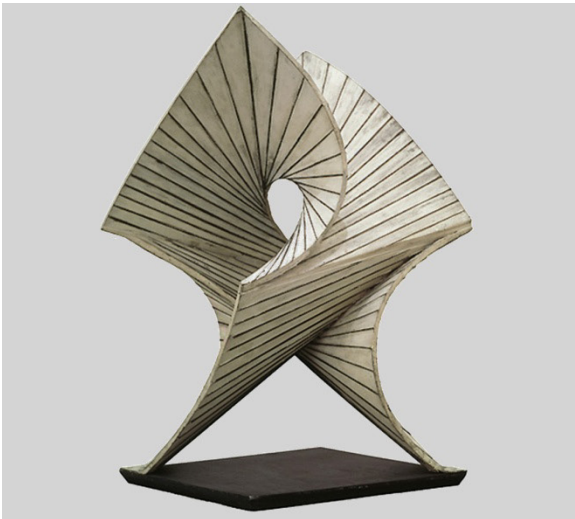

Fig. 4. Tangential developable. On the left: Antoine Pevsner, Developable Surface, 1938; on the right: Antoine Pevsner, Developable Column, 1942.
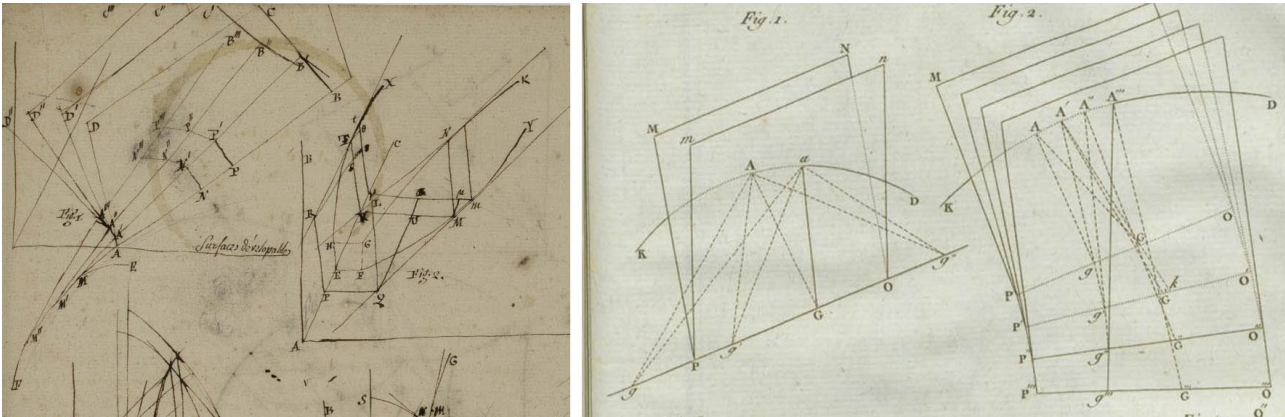
Fig. 5. On the left: Gaspard Monge, 1780 [Monge 1780]; on the right: Gaspard Monge, 1801 [Monge 1801].

surfaces, which are obviously all ruled surfaces, and they can be grouped into three families: tangential developables, conical surfaces and cylindrical surfaces. In a general discussion, the tangential developable can be considered the generic case, in which the directrix (edge of regression) is a generic space curve (fig. 2c). If the edge of regression is a point, we obtain a conical surface (fig. 2a), while if the regression edge is a point at infinity, we obtain a cylindrical surface (fig. 2b).

It is very important to define the concept of unrolling to study how to construct unrolled shapes. As we know, a surface can be unrolled if it can be put on a plane using an isometric transformation, without cuts or overlaps. We can immediately verify this property for the conical surfaces and the cylindrical surfaces (fig. 3), but this is more complex for a tangential developable (fig. 4), and it is complex to have its unrolled shape.

These attempts have been made to find a solution to the question posed by Euler, "quorum superficiem itidem in planum explicare liceat nec ne?", which surfaces can be unrolled on the plane and which cannot? In this way we define the geometric rule that will allow us to move from theory to practice: what can we unroll and how can we find the unrolled shape of a tangential developable? It is known that a ruled surfaces can be unrolled if two generatrices infinitely close intersect each other and they are therefore coplanar. This is clear for the conical surfaces in which all the generatrices pass through a fixed point, the vertex (fig. 2a), and for the cylindrical surfaces because all the generatrices are parallel and then intersect each others in a point at infinity (fig. 2b). In the case of the tangential developable we must use the principles of differential geometry and the concept of limit and derivative to demonstrate this concept. In fact, the tangent of a plane curve at P is the limit potion of PQ line when point Q approximates or tends to P. Defining the tangent at a point P to a plane curve, it is possible to prove that it is unique, so Leibniz introduced the concept of curvature and the definition of osculating circle [Migliari 2009, p. 103]. The osculating circle of a sufficiently smooth plane curve at a given point P on the curve has been traditionally defined as the circle passing through P and a pair of additional points on the curve close to P. The osculating plane to a space curve at a point P of that curve is the plane given by the tangent at P and a neighbour point on the curve. We can demonstrate that tangents of a space curve are intersections of consecutive osculating planes, so developable surfaces can also be defined as the envelope of the movement of osculating plane in space [Fallavolita 2008, p. 111].

Thus, summing up, from a theoretical point of view, the assumptions placed at the base of the experimentation, which is described here, can be summarized as follows:
- all developable surfaces are ruled surfaces;
- all developable surfaces are zero Gaussian curvature;

- developable surfaces can be generated by tangent line motion on a space curve;
- developable surfaces can be generated by osculating plane motion on a spatial curve;
- a surface is unrolled if two successive generatrices are always incidence.

All studies done in the past, whatever is the prevailing approach used (synthetic, analytical or differential geometrical) have historically been based on spatial intuitions that are often poorly represented or not represented at all, and therefore not very "visible". The aim of our research is also to use 3D modeling tools like a method to demonstrate (as well as to show) and to use the generative algorithmic modeling to compare different souls of Geometry: descriptive, analytical and differential.

Therefore, starting from these assumptions, our experimentation is based on hybridization of old principles and traditional methodologies with new generative modeling tools. We are trying to identify innovative research approaches in applied geometry, in which geometry knowledge is always foundation for the solution of complex construction problems.

Methods

Geometric construction of a conical or cylindrical surface does not present particular problems, assigned the directrix and the vertex V (also a point at infinity) we have to construct n generatrices that join the vertex V with the n points of the directrix. You can easily generate the surface using a 3D modeling software, extruding the curve in one direction or to a point (fig. 2). It's much more complex generating tangential developables. It's the same to find unrolled shape, it is very easy to find conical or cylindrical surfaces development using traditional methods or digital tools, on the contrary it is very difficult to find the unrolled shape of a tangential developable.

Tangential developable
In Descriptive Geometry a tangential developable is generated from a spatial, set as the surface swept out by the tangent lines to the curve. This particular group of ruled surfaces can be generated using only one directrix (the edge of regression) because the tangent in P is unique and it is always uniquely determined [Migliari 2009, p. 160].

A tangential developable specializes if the edge of regression is a cylindrical helix: the surface generated by the motion of a tangent line to a cylindrical helix is a developable helicoid.

It is difficult to construct a tangent line to a spatial curve with traditional graphic methods, for this reason, most of historical texts analyzed are without images that could be necessary to show the complex spatial reasonings. The advances in applied geometry derive from the use of digital tools that allow you to automatically construct a tangent line at a point in a spatial curve. To generate the surface, first we have to construct n tangents which, although automatically shown, must be determined one by one, and then the surface can be generated, considering the n generatrices represented. In this way the difference between the generated surface and the tangential development depends on the number of generatrices that you used.

Using the Gaussian curvature analysis tool k, you can verify if the surface thus obtained is developable (green, $k = 0$) or not (blue, $k < 0$) (fig. 6).

The case of the developable helicoid is the simplest, in fact, if the edge of regression is a cylindrical helix, in order to generate the surface it will be sufficient to construct the tangent at a point P and then make it move along the helix [6].

In this case, generative modeling is a powerful tool, useful not only for reiterating procedures but also for verifying theories. In fact, a tangential developable can be unrolled with some unavoidable approximations, as the two consecutive generatrices intersect each others on the edge of regression only in an infinitesimal neighborhood, with n tending to infinity. As part of our experimentation we have developed a definition, that follows geometrical principles, to construct developable ruled surfaces using a general spatial curve. This spatial curve can be imported by Rhino or parametrized in relation to specific needs. Dividing the assigned spatial curve (the edge of regression) in n parts, our algorithmic definition allows to generate the surface by constructing n lines (generatrix of the surface) passing through the n points and tangent to it. By modifying the length of the generatrix and the edge of regression it is possible to obtain infinite developable surfaces. This surface may be cut to define the edge, which is otherwise automatically generated as a function of the generatrix length (fig. 6).

Developable strip modeling using two directrices
Developable strips modeling using two directrices, *c* and *c'* is more usual than tangential developables.
Using the methods of Descriptive Geometry, we know how to model a developable surface using two spatial curves *c* and *c'* (directrices). We have to select a point *P* on the curve *c* and we have to construct the conical surface (vertex *P* and directrix *c'*): the generatrix of the developable surface, in point *P*, is the line obtained by considering the plane passing through *P* and tangent to the conical surface. Reiterating this process we can generate *n* generatrices of the

surface. In this way we have shown that the tangent plane for a generatrix of a developable surface is only one [Migliari 2009, pp. 213-218]. This very important geometric property of the surfaces that can be developed has been fundamental to define the algorithm that allows us to construct a developable strip using two directrices.
However, it is not always possible to obtain a developable surface by arbitrarily assigning the two curves. In fact, it could be that the two curves, arbitrarily assigned, are not "extended" enough, or, vice versa, are too long. For this reason, it is not possible to determine the generatrix in

Fig. 6. Tangential, parametric model: approximation improves (k = 0) increasing the number of generatrices (definition by the author).
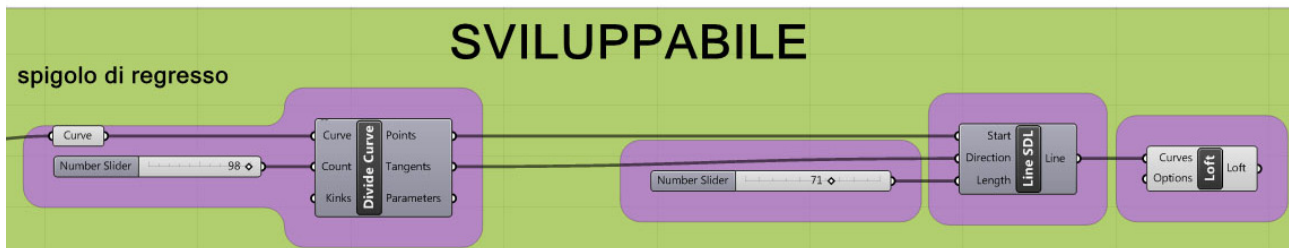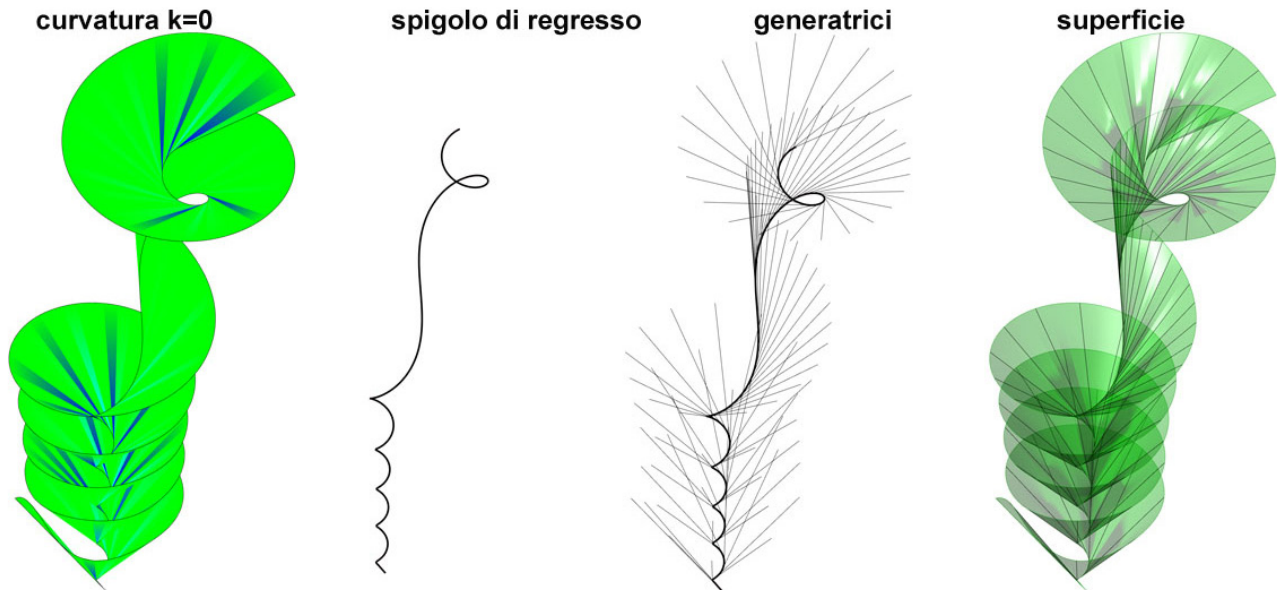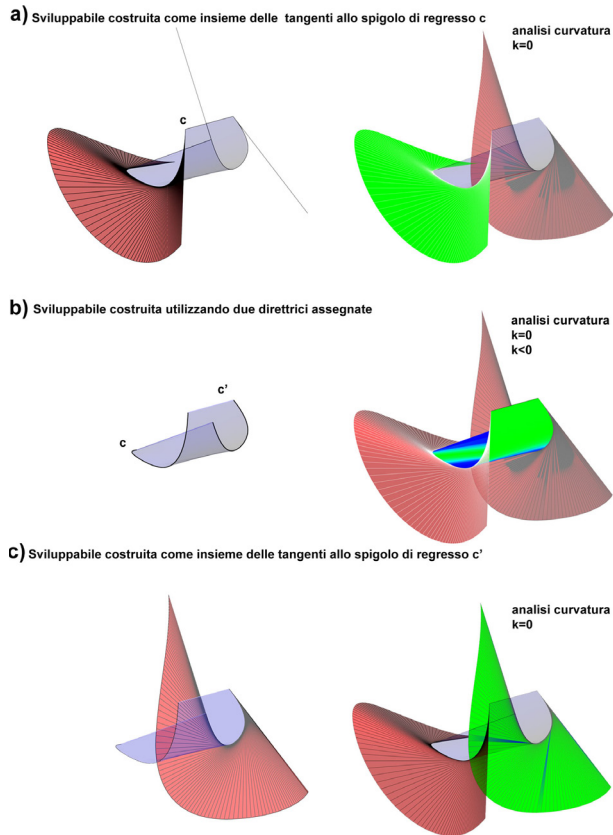
Fig. 7. Developable strips constructed using two spatial curves, c and c '.
Definition of existence domain (graphic elaboration by the author).

a) Sviluppabile costruita come insieme delle tangenti allo spigolo di regresso c

analisi curvatura
k=0

b) Sviluppabile costruita utilizzando due direttrici assegnate

analisi curvatura
k=0
k<0

c) Sviluppabile costruita come insieme delle tangenti allo spigolo di regresso c'

analisi curvatura
k=0

some points. In this case it will be necessary to define the domain of existence of the surface (fig. 7).

The problem seems to be immediately solved using the 3D modeling tools, in fact, the *DevLoft* command of Rhino allows you to automatically generate a developable surface by two spatial curves. This surface has generally zero Gaussian curvature even if in some points it has negative Gaussian curvature (fig. 7), therefore, theoretically, we can't unroll it.

There are some different solutions to solve this problems, it depends on the specific application. One of the possible solutions is certainly to extend the curves

and build the surface by analyzing the curvature. In this way, step by step, by correcting the curves to obtain surfaces with zero Gaussian curvature, you can modeling strips that can be developed, which can be cut according to need.

Through the algorithmic modeling, there are definitions that allow to modify the directices in order to guarantee the existence of the developable.

In our research, we have done the following tests using generative algorithm modeling:
- construction of a developable surface using two directrices, c and c' [7];
- determination of the the edge of regression by joining the consecutive generatrices;
- tangential developable modeling, using our definition;
- comparison of the two surfaces (fig. 8).

This procedure can be used to design a developable strip, because it is patch extracted from tangential developable, and to find the unrolled shape (fig. 9) [Lanzara 2015, pp. 199-203].
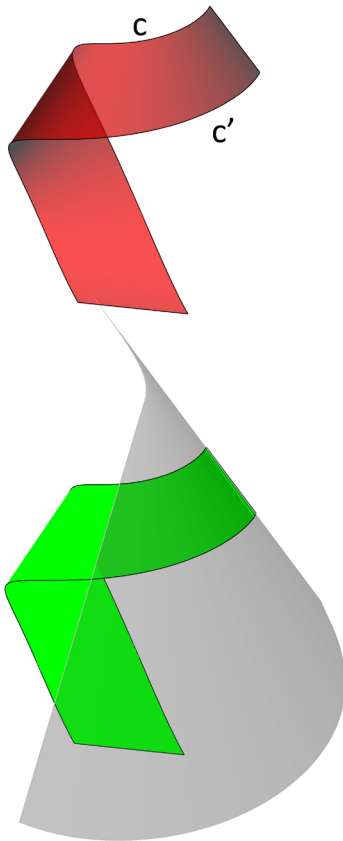
Unrolled shape of tangential developable
A surface, as we said, is developable if it can be unrolled on a plane with rigid movements (isometric transformation), without stretching or tearing: this is possible if two consecutive generatrices are coplanar.

We have analyzed principles and methods used to find the unrolled shape of a tangential developable. We have defined different approaches that allow us to determine the "approximate unrolled shape" of a non-developable surface. We have also considered materials and manufacturing techniques that can solve approximation problems connected to "smash", allowing the panel deformation thanks to "cuts", *kerfing*, or "overlapping", *bending*.

As we noted, developable surfaces have zero Gaussian curvature, consequently they can be manufactured, using a flexible and non-deformable material, starting from their unrolled shape, simply by shaping the cut out shape. This kind of surfaces are easy to manufacture and this has favored their diffusion.

In the case of cylindrical surfaces the edge of regression is a direction, so the generatrices are all incidents in one point at infinity. Using the traditional method introduced by Monge, in order to determine the development of the surface, the directrix is divided into *n* parts and a prismatic surface is obtained. This coincides with the conical surface when *n* to infinity. Cylindrical surface is
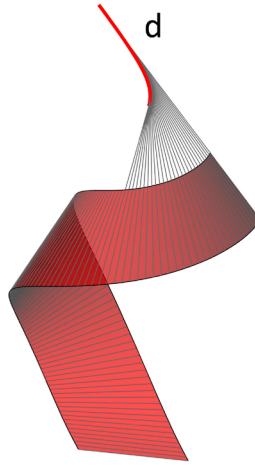
Fig. 8 Algorithmic modeling: comparison between strip obtained using two directrices and the tangential generating using the edge of regression.

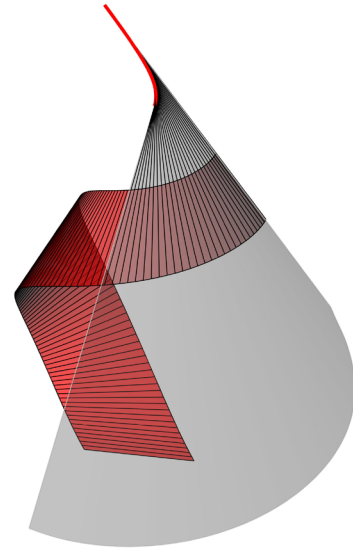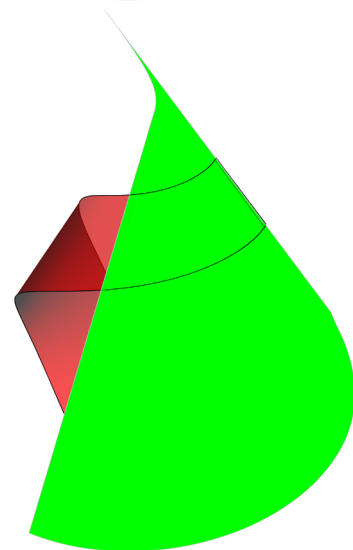**1. striscia su due direttrici c e c'**   **2. costruzione spigolo di regresso d**   **3. tangenziale su spigolo di regresso**
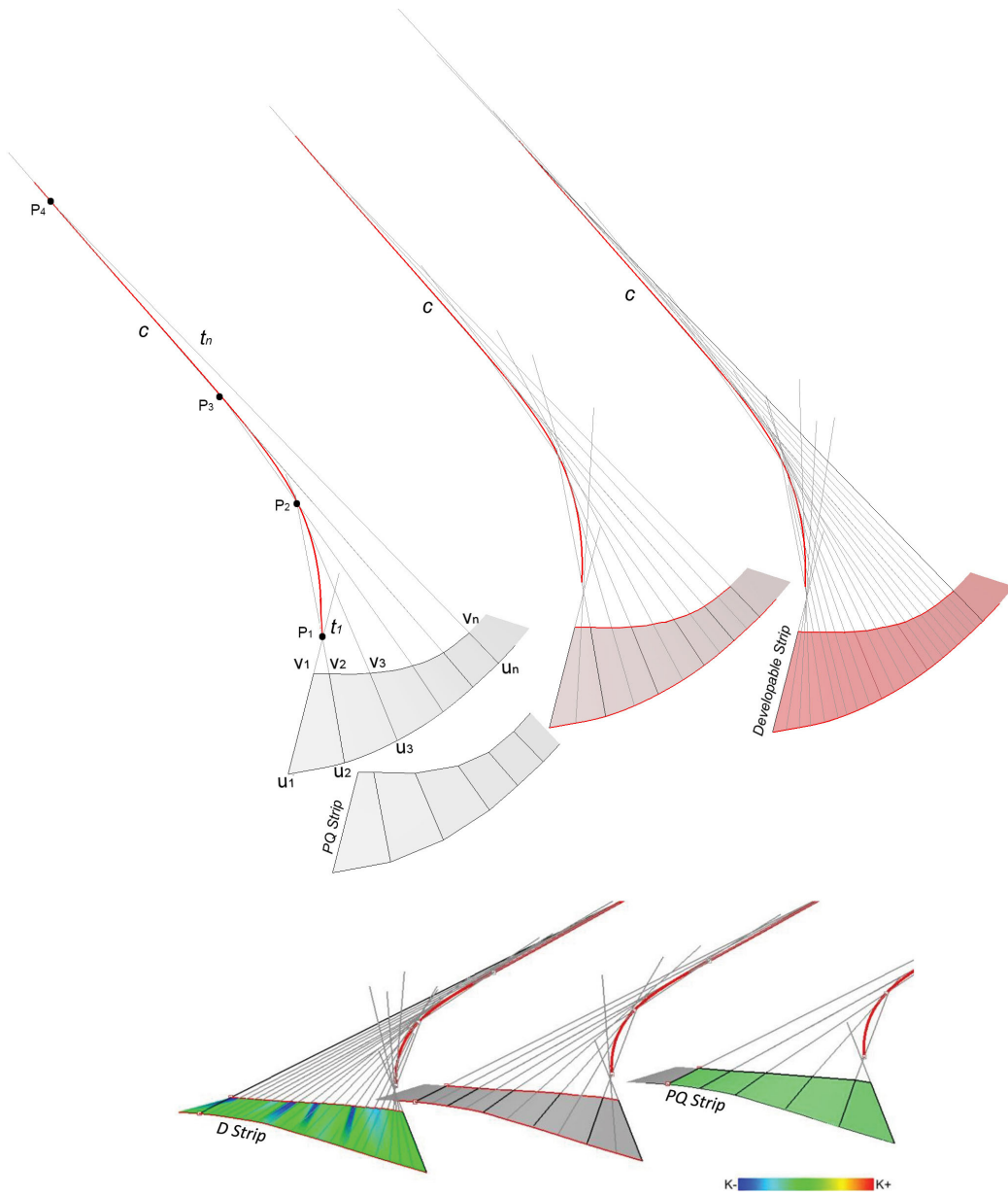

4. analisi della curvatura gaussiana

K=0 - curvatura della striscia

K=0 - curvatura della tangenziale

Fig. 9. Local optimization approaches to design a developable strip (graphic elaboration by Emanuela Lanzara).

developed by unrolling the *n* quadrilateral faces in sequence on a plane.

Furthermore, conical surfaces can be considered developable surfaces in which the edge of regression is a point, so two consecutive generatrices are always intersecting lines. To define the unrolled surface, we divide the directrix into n parts and transform the continuous surface into a discrete surface: a pyramid. using 3D modeling, there is a command that is able to automatically unroll both conical surfaces and cylindrical surfaces [8].

The method for finding the unrolled shape of a tangential developable is more complex, in this case differential geometry application is evident. Monge uses the principles of differential calculus to study the properties of the developables surfaces [Migliari 2009, pp. 106-108]. Each developable surface can be flattened onto a plane without distortion and, in a limited region, without overlapping. The unrolled shape of the surface generated by the infinite tangents to a space curve is obtained by considering n generatrices and flattening onto plane the surfaces included between two consecutive generatrices. If we consider two consecutive tangents $t_1 t_2$ (fig. 10), in theory incidents, they identify a plane, so if we rotate $t_2 t_3$ around t_2 and repeat the operation for the subsequent tangents we find the unrolled surface. The unrolled surface depends on the edge of regression. It may happen that the configuration of the surface is such that portions of unrolled surface overlap with the others [Fallavolita 2008, p. 113], in these cases, it is necessary to divide the design surface into parts in order to manufacturing it.

Development is therefore easy with regard to conical and cylindrical surfaces, in this regard the Sereni says: "*il metodo rigoroso non può che desumersi dal calcolo ed i metodi approssimativi sono essi medesimi soverchiamente lunghi, e di sì raro uso nelle arti che no meritano d'arrestarci davvantaggio… in ultima analisi tutti si ridurrebbero a costruire lo sviluppo di una superficie poliedrica… e quanto minore fossero gli angoli tanto più il lavoro si accosterebbe alla precisione*" [Sereni 1826, p. 49] [9].

Interesting is the approach of Leroy that highlights the importance of result knowledge to distribute errors. The scholar addresses the issue of unrolled shape of tangential developable, in particular of the developable helicoid, in the same way for a conical or cylindrical surface, he says: "*dividendo una curva piana situata sulla superficie in piccoli archi sensibilmente confusi con le loro corde: allora i settori elementari proiettati potranno essere considerati come triangoli di maniera che, se si costruiscono questi triangoli sopra uno stesso piano ed allato gli uni degli altri, il loro insieme rappresenterà lo sviluppo della superficie in questione*" [Leroy 1826, p. 289] (fig. 10). Leroy underlines that the need to approximate a continuous surface into polyhedric surface results in an accumulation of errors that could be avoided if we could know the unrolled shape of the curve. We know that helixes on developable helicoid turn in concentric circles, for this reason the edge of regression will turn into a circle whose radius depends on the radius of curvature of the helix ($O_2 A_2$) and it can be determined by using the differential calculation or graphically. To draw the unrolled shape of the developable helicoid, it will be sufficient to fix the length of the assigned generatrix (for example $A_2 W_2$) on the helix development and draw a concentric circle with radius $O_2 W_2$ (fig.10).

We have defined a method that allows to find the unrolled shape of any developable tangential using the algorithmic modeling. If we divide the edge of regression into *n* parts and we consider *n* tangents (generatrices of ruled surface) we have that two consecutive tangents intersect on the edge of regression. This is true only in a small, infinitesimal neighborhood. In fact, if we divide the edge of regression into n parts and consider two successive tangents, t_1 and t_2, led respectively by points *1* and *2* (fig. 10), we define the non-flat quadrilateral *A12B*. When point 2 goes to point *1*, points *1*, *2* and *B* can be considered aligned, it follows that, approximately, it will always be possible to define a flat triangular face and then unroll the surface composed by *n* triangular faces. The approximation of the unrolled surface obviously depends on *n*.

We have done two tests in our research and we have analyzed the results to evaluate which of the two methods allows to obtain the unrolled shape that best approximates the real surface. In the first case we have divided the surface using the *n* tangents and we have defined a surface composed by ruled surfaces obtained by using the consecutive tangents, *A1*, *B2*, *C3*… (fig. 10). These ruled surfaces are modeled by using the no-flat quadrilater *A12B*, therefore, as we have previously said, they cannot be unrolled. Therefore, they have been "flattened" using the *smash* tool, which, using Rhino, allows us to determine the approximate development of
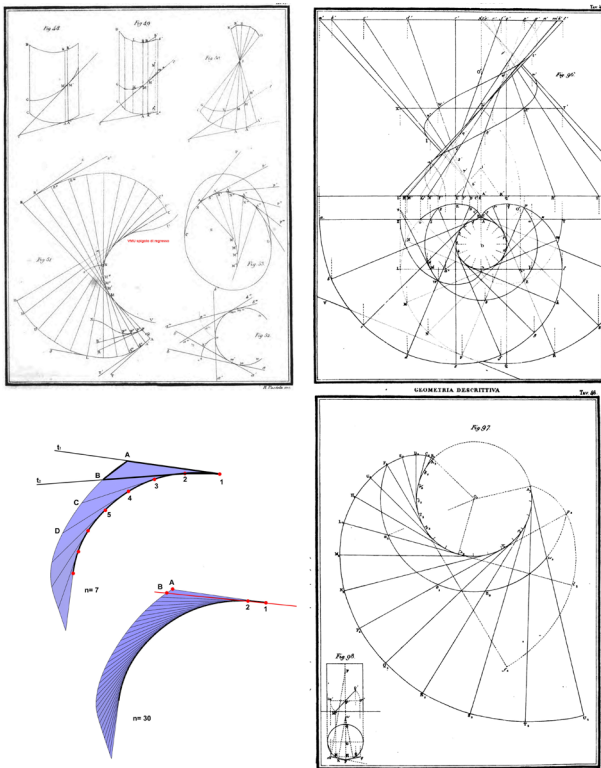
Fig. 10. Charles François Antoine Leroy: generic developable, developable helicoid, development of helicoid. Division in twisted quadrilateral or in triangular elements.

a non-developable surface. We have determined the unrolled shape of the surface by sequentially unrolling the quadrilaterals onto plane [10].

In the second case, we have supposed that the points 1, 2, B, and 2, 3, C are aligned and we have split the surface into triangles AB1, BC2, CD3… and we have determined the unrolled shape of the surface by unrolling the triangles onto plane. Comparing the metric values of the 3D surface, namely the length of the edges and the area, with the unrolled shapes that we have constructed, it results that, in the first case, the unrolled surface is larger than the real one, while in the second case it is smaller. For this reason, in order to construct exactly the 3D shape, using the unrolled

shape that we have determined, in the first case overlaps (bending) must be provided, in the second case we have to cut (kerfing) and the material must be "deformable".

Approximate flattened shape of double curvature surface
There is no doubt about the advantages offered by using developable surfaces to manufacture objects that can be fabricate using a flat panel. To do that you need to know the unrolled shape in order to draw the exact contour of the surface to be cut out on the plane. We can find approximate but sufficiently precise unrolled shape of non-developable surfaces useful for certain applications. We have identified in our research the following most significant approaches to construct a double curved surface using flat elements:
1. to approximate the complex surface splitting it into strips that can be developed, then identifying some remarkable lines on the surface in order to optimize the construction process;
2. to design the surface using strips that can be developed [Liu et al. 2006];
3. to use processes that make the panel flexible and deformable (kerfing or bending) to manufacture the shape designed from a flat element.
We have studied non-developable surfaces and in particular the case of hyperbolic paraboloid, to highlight some of the problems and to define some possible approaches to transform a non-developable surface into a flat surface that, with better approximation, is able to preserve the characteristics of the 3D surface.
Main research goal is to highlight, through the applications, how these approaches can influence the figurative outcome and the manufacturing process.
The hyperbolic paraboloid is a ruled surface that may be generated by a moving line that is parallel to a fixed plane, it is a not developable surface because two consecutive generatrices are always skew lines and Gaussian curvature is always negative.
There are several tools that allow you to automatically obtain the approximate unrolled shape of a non-developable surface: using Rhino the command *smash* and the command *squish* (fig. 12). The critical analysis of the results obtained using a 3D modeling software is part of our experimentation. Using the *smash* command we can automatically generate an approximate unrolled shape for a double curved surface, but using this flat shape we can reconstruct the real 3D shape only if we use a deformable

Fig.11. *Procedural modeling: tests for developable helicoid fabrication (graphic elaboration by the author).*

eliche cilindriche

spigolo di regresso

tangenti generatrici

superficie griglia

distribuzione pattern su superficie

SVILUPPABILE

spigolo di regresso

tangenti ◇ 46

Curve · Points · Count · Tangents · Kinks · Parameters · Divide Curve

lunghezza   200 ◇

Start · Direction · Length · Line SDL · Line

Curves · Options · Loft · Loft

inserire spigolo di regresso per generare una sviluppabile tangenziale

A

matrice pattern

A

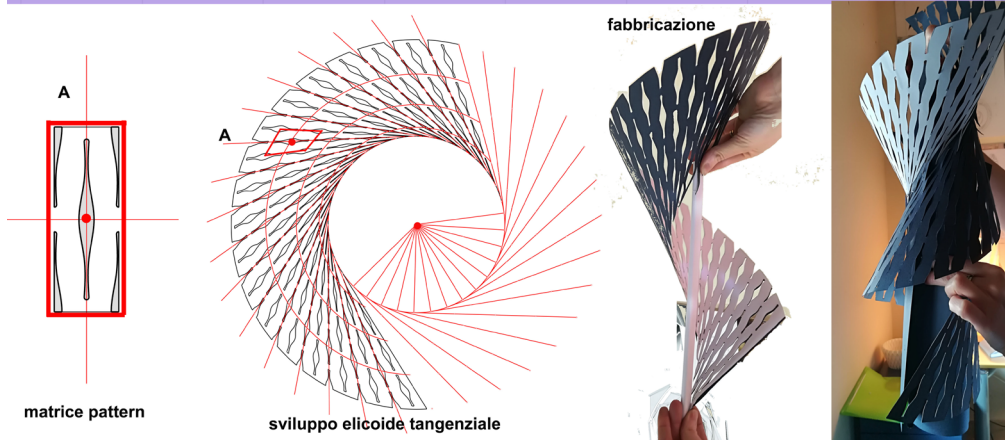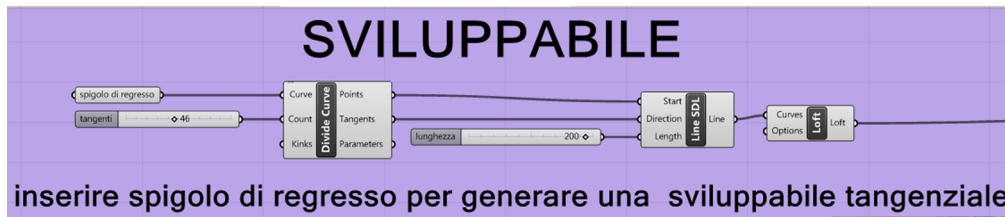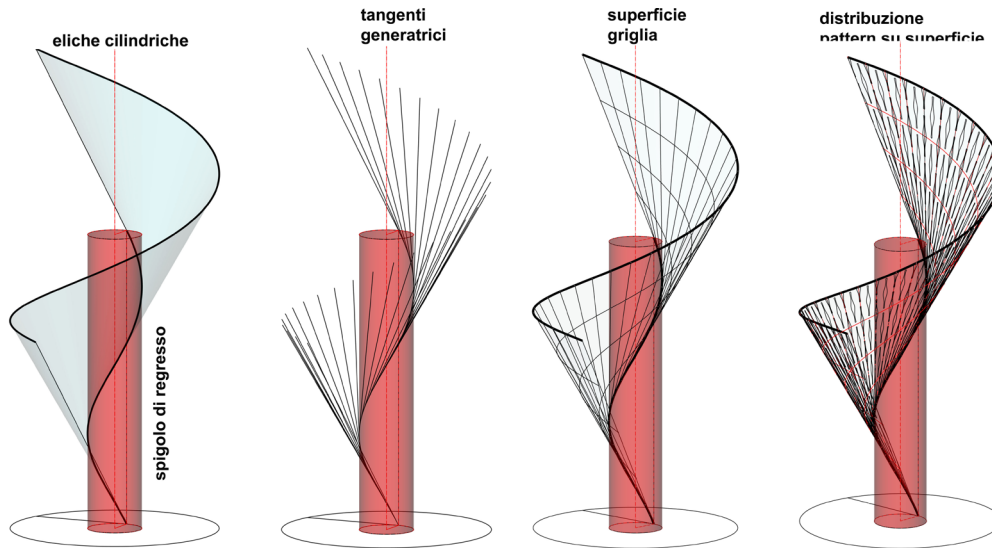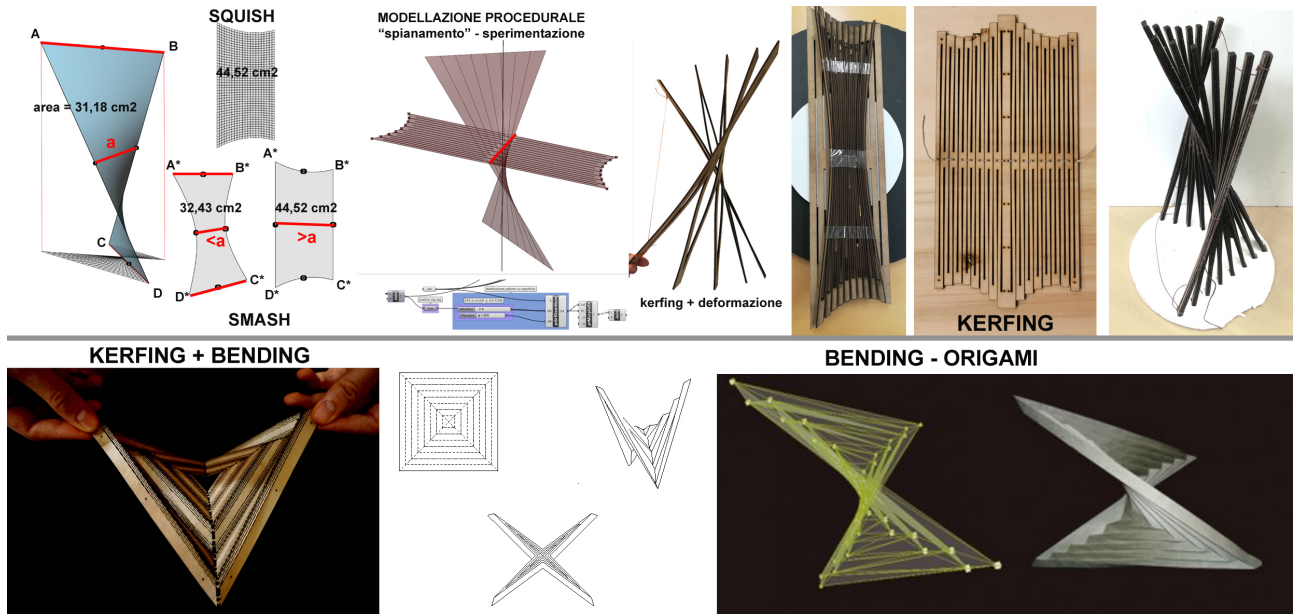sviluppo elicoide tangenziale

fabbricazione

Fig. 12. Prototypes in wood and cardboard for a lamp manufacture: portions of developable helicoids. Design by Mara Capone.

Fig. 13. Tests for hyperbolic paraboloid manufacture: kerfing and bending experiments.

material. The *squish* command uses a different algorithm, performs the smoothing of meshes or 3D NURBS surfaces, modifying the starting area, allowing the display and control of the local compression and stretching zones.

Applying the *smash* and *squish* commands to the hyperbolic paraboloid piece, used in our tests, we obtained different shape (fig 12). We have done these observations based on results: the area changes with respect to the real one and the generatrices of one of the two groups deform themselves, it follows that to transform the flatten shape into 3D designed shape it will necessarily be breakings and/or overlaps. In fact, if the generatrix *AD* becomes curve, it turns into the curved edge *A'D'*, this must be deformable, therefore the cuts must be made to allow the curve *A'D'* to assume the configuration straight of the designed shape. Similarly, if the generatrix *AB* is deformed, it will be necessary to allow that the curve *A'B* is able to be transformed into the straight segment *AB* (fig. 12).

Using generative modeling we tested different methodologies to simulate the deformation according to

the cuts made. Our goal is to identify processes and to develop tools to define the approximate flatten shape of a double curvature based on the knowledge of geometric properties.

Conclusions and future research developments

The topic of fabricating 3D complex surface shape using a flat surface has been historically addressed and it is the basis of search for optimized solutions based on applied geometry. The use of parametric modeling tools allows us to address this very complex problem, opening new fields of experimentation and research based on ancient principles whose verification is always better supported by diffusion of digital manufacturing techniques.

Our research starts from the study of geometry and algorithmic modeling tools and, by hybridizing different methodologies, tends to develop general solutions that can be used in different fields.

Notes

[1] Hachette proposes a classification of surfaces in three groups: developable surfaces, surfaces of revolution and ruled surfaces "*de surfaces qu'onvient de définir et qui sont désignées par le noms de surfaces developpables, surfaces de révolution, surfaces réglées*" [Hachette 1828, p. 30].

[2] We know that the *osculating circle* is the circle that approaches the curve most tightly in an infinitesimal interval. The curvature at one point *P* is the inverse of the radius of the osculating circle $k=1/r$. The osculating plane in a point *P* of a space curve, is the limit position taken by the plane passing through the tangent in *P* to the curve and for another point *Q* of the curve, to the tendency of *Q* to *P*. If we consider a point *P* of a skewed curve, the osculating plane is the plane identified by the tangent vector *t* in *P* and by the normal vector *n*.

[3] "*Notissima est proprietas cylindri et coni, qua eorum superficiem in planum explicare licet atque ad eo haec proprietas ad omnia corpora cylindrical et conica extenditur, quorum bases figuram habeant quamcunque; contra vero sphaera hac proprietate destituitur, quum eius superficies nullo modo in planum explicari neque superficie plana obduciqueat; ex quo nascitur quaestio aeque curiosa acnotatudigna, utrum praeterconos et cylindros alia quoque corporum genera existant, quorum superficiem itidem in planum explicare liceat nec ne? Quamob rem in hac dsisertationes equens considerare constitui Problema: Invenire aequationem generalem pro omnibus solidis, quorum superficiem in planum explicare licet, cuius solutionem variis modis sum agressurus*", Euler 1772.

[4] The most advanced researches in complex surfaces manufacture take place in the field of applied geometry. One of the possible solutions is to divide the surface into parts that can be made by approximation using

strips that can be developed. This process is very advantageous for surface manufacture that can be built using flat elements to be put into shape.

[5] In his first lessons of Descriptive Geometry, Gaspard Monge teaches an elegant "existential demonstration" of ruled surfaces, generated by the motion of a straight line that is supported by three generic spatial curves assumed as direcrices. [Migliari 2009, p. 154].

[6] For example, using Rhinoceros, you can use *sweep one rail*.

[7] We used *Tapeworm script by Mårten Nettelbladt* to generate a developable strip. The tool allows to modify two directrices in order to guarantee the existence of the developable.

[8] Using Unroll, Rhinoceros allows to determinate the development of developable surface. We can automatically unroll conical and cylindrical surfaces. There are some problems to determine development of a tangential surface, even if it is developable.

[9] Sereni is a supporter of analytical method, in fact, he states that "the approximate methods are themselves overwhelmingly long, and so rarely used in the arts that they do not deserve to be arrested, they would ultimately […] reduce to building" [Sereni 1826, p. 49].

[10] Using generative algorithms the development of the tangential ruled surface was determined by dividing into n parts (variables) the regression edge and building the tangents to the curve passing through the n points so determined. In this way, the surface generated by the subsequent tangents was determined.

Author

Mara Capone, DiArc Department of Architecture, University of Naples *Federico II*, mara.capone@unina.it

Reference List

Euler, L. (1772). De solidis quorum superficiem in planum explicare licet. In *Novi commentarii academiae scientiarum imperiatis Petropolitanae*. XVI. pp. 3-34.

Fallavollita, F. (2008). *Le superfici rigate e le superfici sviluppabili. Una rilettura attraverso il laboratorio virtuale*. Tesi di dottorato di ricerca in Scienze della Rappresentazione e del Rilievo. Tutor: L. De Carlo, R. Migliari. Sapienza Università di Roma, Dipartimento di Rilievo, Analisi, Disegno dell'Ambiente e dell'Architettura.

Gauss, K.F. (1902). *General Investigations of Curved Surfaces of 1827 and 1825, translated with notes and a bibliography by James Caddal Moreheas and Adam Miller Hiltebeitel*. Princeton: The Princeton University Library.

Hachette, J.N.P. (1828). *Traité de Géométrie Descriptive*. Paris: Corby.

Lanzara, E. (2015). *Paneling Complex Surfaces. Razionalizzazione di superfici complesse per l'industrializzazione*. Doctoral thesis in Tecnologia dell'Architettura e rilievo e rappresentazione dell'architettura e dell'ambiente. Tutor: M. Capone, cotutor: A. Picerno Ceraso. Università degli Studi di Napoli *Federico II*.

Leroy, C.F.A. (1846). *Trattato di Geometria Descrittiva con una collezione di disegni composta da 60 tavole*. French translation with notes by Salvatore d'Ayala e Paolo Tucci, Napoli: Dalla Reale Tipografia della Guerra. (First ed. *Traité de géométrie descriptive*, Parigi 1842).

Liu, Y. et al. (2006). Geometric Modeling with Conical Meshes and Developable Surfaces. In *ACM Transactions on Graphics*, 25, 3, pp. 681-689.

Migliari, R. (2009). *Geometria descrittiva*. Roma: CittàStudi.

Monge, G. (1780). Mémoire sur les propriétés de plusieurs genres de surfaces courbes et particulièrement sur celles des surfaces développables avec une application à la théorie générale des ombres et des pénombres. En *Mém. div. Sav.*, t. 9, pp. 382-440.

Monge, G. (1785). *Mémoire sur les développées, les rayons de courbure, et les différens genres d'inflexions des courbes a double courbure*. Paris: De l'Imprimerie Royale.

Monge, G. (1798). *Géométrie Descriptive*. Paris: Baudouin.

Monge, G. (1801). *Feuilles d'analyse appliquée à la géométrie, appliquée à la géométrie à l'usage de l'Ecole Polytechnique*. Paris: Baudouin.

Quintial, F.G., Barrallo, J., Artiz Elkarte, A. (2015). Freeform surfaces adaptation using developable strips and planar quadrilateral facets. In *Journal of Facade Design and Engineering*, 3, pp. 59-70.

Sereni, C. (1826). *Trattato di Geometria Descrittiva*. Roma: Nella Stamperia di Filippo e Nicola De Romanis.

Snežana, L. (2011). Developable Surfaces: Their History and Application, In *Nexus Network Journal*, 13, pp. 701-714.

Tang, C., Bo, P., Wallner, J., Pottmann, H. (2016). Interactive design of developable surfaces. In *ACM Transactions on Graphics*, 35, 2, Article 12: <http://www.geometrie.tugraz.at/wallner/abw.pdf> (accessed 2018, October 23).